ASSIGNMENT 7

1

(a) A discrete time dynamical system is of the form

m_0 = given initial value/condition

$m_{t+1} = f(m_t)$ ... rule

where t is time and 1 in $t+1$ represents one unit of time. m_t is called input, m_{t+1} is the output. The updating function is the rule which states how output is obtained from input

(b) The sequence of numbers $m_0, m_1 = f(m_0), m_2 = f(m_1), \ldots$ is the solution

(c) It is given that $M_0 = 4$ and $M_{t+1} = 0.5M_t + 1$

$$M_1 = 0.5(4) + 1 = 3$$
$$M_2 = 0.5(3) + 1 = 2.5$$
$$M_3 = 0.5(2.5) + 1 = 2.25$$
$$M_4 = 0.5(2.25) + 1 = 2.125$$
$$M_5 = 0.5(2.125) + 1 = 2.0625$$

(d) $p_{t+1} = 0.57 p_t \longrightarrow p_t = p_0\, 0.57^t = 12 \cdot 0.57^t$

so $p_{100} = 12 \cdot 0.57^{100} \approx 4.64 \cdot 10^{-24}$

(e) $p_{t+1} = p_t + 0.57 \longrightarrow p_t = p_0 + 0.57t = 12 + 0.57t$

so $p_{100} = 12 + 0.57(100) = 69$

2. (a) N_t = # of deer at time t (in years)

$$N_{t+1} = N_t + 0.045N_t = 1.045N_t\,; \quad N_0 = 120$$

(b) B_t = # of bacteria at time t (in hours)

$$B_{t+1} = 3B_t - 1{,}000\,; \quad B_0 = 3{,}000$$

(c) B_t = # of bacteria at time t (in hours)

$$B_{t+1} = 3(B_t - 1{,}000)\,; \quad B_0 = 3{,}000$$

(d) H_t = height of a tree at time t (in years)

$$H_{t+1} = H_t + 4\,; \quad H_0 = 1.5$$

(e) M_t = amount of medication in patient's body at time t (in hours)

$$M_{t+1} = \underbrace{0.7M_t}_{\text{30\% absorbed, so 70\% left}} + 0.5\,; \quad M_0 = 2$$

3. (a) $f(x_t) = 2x_t + 30$

$\quad\;\; \overset{\shortparallel}{x_{t+1}}$

domain: number of mites at time t

range: number of mites at time $t+1$

(b) domain: amount of medication left in patient's body at time t

range: amount ..... left ... at time $t+1$

(c) L_t = height in inches

$$L_{t+1} = 39.37\, l_{t+1} \qquad 1\,m = 39.37\,in$$
$$= 39.37(1.1 l_t + 0.2)$$
$$= 1.1 \cdot 39.37 l_t + 39.37 \cdot 0.2$$
$$= 1.1 L_t + 7.874$$

$$L_0 = 1.2(39.37) = 47.244$$

(d) To obtain the solution of a DS (dynamical system) geometrically, using the graph of the updating function

4.(a) The value(s) that does/do not change under the system; or an equilibrium is the intersection of the updating function and the diagonal

(b) when $l_t = 3$, $l_{t+1} = 1.1(3) + 0.2 = 3.5 \neq 3 = l_t$

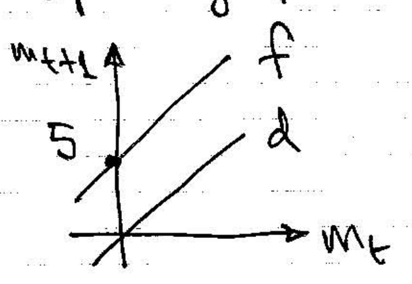
(c) when $m_t = \underline{\underline{4}}$, $m_{t+1} = 0.25(4) + 3 = 1 + 3 = \underline{\underline{4}} = m_t$

(d) $m_{t+1} = m_t + 5$

equilibrium: $m^* = m^* + 5$

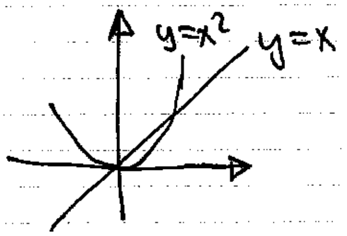
$0 = 5 \rightarrow$ no solutions

or: updating function $f(m_t) = m_t + 5$



does not intersect the diagonal $m_{t+1} = m_t$

(e) We need a graph which intersects the diagonal at two points, for example:



so $y=x^2$ is updating function:

$$\begin{cases} M_{t+1} = M_t^2 \\ M_0 = \text{any value} \end{cases}$$

equilibrium

$$M^* = M^{*2}$$
$$M^* - M^{*2} = 0$$
$$M^*(1-M^*) = 0 \rightarrow M^* = 0, 1$$

5. $-0.6 \rightarrow -0.9 \ldots \quad n_{t+1} = -0.9 n_t + 5.3$

$$n^* = -0.9 n^* + 5.3$$
$$1.9 n^* = 5.3 \rightarrow n^* = 5.3/1.9$$

so there <u>is</u> an equilibrium

$-0.6 \rightarrow 1.3 \quad \text{-----} \quad n_{t+1} = 1.3 n_t + 5.3$

$$n^* = 1.3 n^* + 5.3$$
$$-0.3 n^* = 5.3$$
$$n^* = 5.3/-0.3 < 0$$

there is an equilibrium, but makes no sense since n^* is the number of cod fish

6. The system describes the absorption of a drug with replacement: each day, α% is absorbed, and 1 unit of drug is added. It turns out that the equilibrium value is the reciprocal of the amount absorbed, $M^* = \frac{1}{\alpha}$

7. (a) Consider $c_{t+1} = 0.87c_t + d$

where $c_0 = 3 \cdot 200 + 80 = 680$ mg

$d = 0$

so the system is

$$\begin{cases} c_{t+1} = 0.87c_t \\ c_0 = 680 \end{cases} \Longrightarrow c_t = \overset{680}{\overset{\parallel}{c_0}} \cdot 0.87^t$$

at midnight, there will be $c_2 = 680 \cdot 0.87^2 \approx 515$ mg of caffeine left in our body

(b) $\boxed{\text{elimination rate is 13\% per hour}}$

so $c_{t+1} = 0.87c_t \longrightarrow c_t = c_0 \cdot 0.87^t$

half life: $0.5c_0 = c_0 \cdot 0.87^t \longrightarrow$ compute t, as on page 123

(c) Consider $c_{t+1} = 0.87c_t + d$

where $c_0 = 200$ mg (c_0 = noon)

$d = 200$ mg

$$\begin{cases} c_{t+1} = 0.87c_t + 200 \\ c_0 = 200 \leftarrow \text{noon} \end{cases}$$

use calculator: $c_1 = 0.87(200) + 200 = 374$ ← 1pm

$c_2 = 525.38$

$c_3 = 657.08$

$c_4 = 771.66$

$c_5 = 871.34$

6 pm $\longrightarrow c_6 = 958.07$

NOTE: answer depends on how you round off intermediate values of c

8. It is the number of offspring produced by a single member of a population

1.05 → each individual, on average, produces 1.05 offspring → population increases

9. Per capita production rate is $\frac{1 \text{ thousand}}{1 \text{ million}} = 0.001$ new members per member per year.

Check:
$$1{,}000 = \underbrace{\text{per capita rate}}_{0.001} \cdot 1{,}000{,}000$$

10. Start with $p_{t+1} = r \cdot p_t$

$p_0 = 5{,}000$ (so $t=0$ ... year 1990)

→ the solution is $\underline{p_t = 5{,}000 \cdot r^t}$

find r: in 2009, the population is 1,900

$t = 19$ $\qquad$ $p_{19} = 1{,}900$

so from $p_{19} = 5{,}000 \cdot r^{19}$ we get

$$1900 = 5{,}000 \cdot r^{19}$$

$$r = \left(\frac{19}{50}\right)^{1/19} \approx 0.95035$$

So $\underline{p_t = 5{,}000 \cdot 0.95035^t}$

now find t so that $p_t = 500$

$$500 = 5{,}000 \cdot 0.95035^t$$

$$0.1 = 0.95035^t \longrightarrow \ln 0.1 = \ln 0.95035^t$$

$$t = \frac{\ln 0.1}{\ln 0.95035} \approx 45.21 \text{ years}$$

11. The consumption of half a drink every hour leads to a decrease of the amount of alcohol; the consumption of one drink/hour increases the amount of alcohol in the system.

12. $$a_{t+1} = a_t - \frac{10.1\,a_t}{4.2 + a_t} + \underset{28}{\underset{\shortparallel}{d}}$$

$a_0 = 0$ ... initially, no alcohol present
then, every hour, two drinks

$a_1 = 0 - 0 + 28 = 28$

Use calculator

$a_2 = 47.22$

$a_3 = 65.94$

$a_4 = 84.44$

$a_5 = 102.82$

13. $$\cancel{a^*} = \cancel{a^*} - \frac{10.1a^*}{4.2 + a^*} + d \longrightarrow \frac{10.1a^*}{4.2+a^*} = d$$

$$10.1\,a^* = 4.2d + a^*d$$

$$(10.1 - d)\,a^* = 4.2d \longrightarrow a^* = \frac{4.2d}{10.1 - d}$$

(keep in mind that $d \geq 0$)

If $d = 10.1$ → denominator is 0, so no equilibrium

If $d > 10.1$ then a^* is negative → mathematically that is ok, but makes no sense biologically

If $d < 10.1$ then there is equilibrium

Thus, if $d \geq 10.1$ there is no (or no meaningful) equilibrium