MATHEMATICS 1LS3 TEST 1

Day Class
Duration of Examination: 60 minutes
McMaster University, 1 October 2013

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FIRST NAME (please print): SOLUTIONS

FAMILY NAME (please print): __________

Student No.: __________

THIS TEST HAS 8 PAGES AND 7 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit.

Problem	Points	Mark
1	6	
2	6	
3	6	
4	5	
5	7	
6	5	
7	5	
TOTAL	40	

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Name:____________________
Student No.: ____________________

1. (a)[3] Which of the following expressions are defined, i.e., are real numbers?

(I) $f(1)$ if $f(x) = \ln(e^x - 4)$ NO $e^1 - 4 < 0$

(II) $f(2)$ if $f(x) = \arcsin x$ NO domain of arcsin x is $[-1,1]$

(III) $f(3)$ if $f(x) = (1 - x^2)^{-1}$ YES denominator $\neq 0$

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

(b)[3] Which of the following statements is/are true for the discrete-time dynamical system $p_{t+1} = 0.23p_t$, $p_0 = 200$?

(I) The solution p_t ~~increases~~ decreases exponentially → $p_t = 200 \cdot 0.23^t$

(II) The updating function is exponential → $f(p_t) = 0.23\, p_t$, linear

(III) The value $p^* = 0$ is an equilibrium

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

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2. **Identify each statement as true or false (circle your choice). No justification is needed.**

(a)[2] If $m_{t+1} = m_t + 12$ and $m_0 = 7$, then $m_{10} = 120$.

TRUE FALSE

$m_t = m_0 + 12t = 7 + 12t$

so $m_t = 7 + 12(10) = 127$

(b)[2] If T is inversely proportional to S, then S is (directly) proportional to T.

TRUE FALSE

If $T = c \cdot \frac{1}{S}$ then (solve for S)

$S = c \cdot \frac{1}{T}$ → also inv. prop.!

(c)[2] The semilog graph of $f(t) = 12e^{-0.4t}$ is a line of negative slope.

TRUE FALSE

$\ln f(t) = \ln 12 - 0.4t$

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Questions 3-7: You must show work to receive full credit.

3. Medical tests show that the half-life of Ibuprofen (nonsteroidal anti-inflammatory drug) taken by adult women is about 1.8 hours. A female patient is given 400 mg of Ibuprofen.

(a)[2] By thinking in terms of half-lives, determine whether, after 6 hours, there is more or less than 50 mg of Ibuprofen left in the patient's body.

$$400 \xrightarrow{1.8} 200 \xrightarrow{1.8} 100 \xrightarrow{1.8} 50 \;\dots\; 25$$

5.4 hours up to here

6 hours

$\boxed{< 50 \text{ mg}}$

(b)[4] Calculate the amount of Ibuprofen left in the patient's body after 6 hours. In your calculations, round off to three decimal places. Make sure that your answer does not contradict part (a).

$$I(t) = I(0)e^{kt} \longrightarrow 0.5\,\cancel{I(0)} = \cancel{I(0)}\,e^{1.8k}$$

so $1.8k = \ln 0.5$

$k = \ln 0.5 / 1.8 \approx -0.385$

thus $\boxed{I(t) = I(0)\, e^{-0.385t}}$

and $I(6) = 400 \cdot e^{-0.385(6)} \approx 39.705$ mg

so it is less than 50 mg

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4. The population of Grizzly bears in the western half of Alberta is estimated to be

$$P(t) = \frac{160}{1 + 1.2e^{-0.4t}}$$

where t is time in years, with $t = 0$ representing the year 1980.

(a)[2] What question is answered by finding the inverse function of $P(t)$?

Given P, find t
i.e., what time (what year) was it when the population was P?

(b)[3] Find the inverse function of $P(t)$.

$$P = \frac{160}{1+1{,}2e^{-0.4t}}$$

$$P + 1{,}2Pe^{-0{,}4t} = 160$$

$$1{,}2Pe^{-0{,}4t} = 160 - P$$

$$e^{-0.4t} = \frac{160-P}{1{,}2P}$$

$$-0.4t = \ln\left(\frac{160-P}{1{,}2P}\right)$$

$$t = -\frac{1}{0{,}4}\ln\left(\frac{160-P}{1{,}2P}\right)$$

or $t = -2{,}5\ln\left(\frac{160-P}{1{,}2P}\right)$

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5. In a study of networks of neurons, researchers use the following intensity functions as generators of oscillatory inputs:

$$\lambda_1(t) = v_0 + a\cos(\pi m t) \quad \text{and} \quad \lambda_2(t) = v_0 + a\cos(\pi m t + d)$$

By reading the paper, you learn that the parameters v_0, a, m and d are all positive.

(a)[1] What is the period of $\lambda_1(t)$?

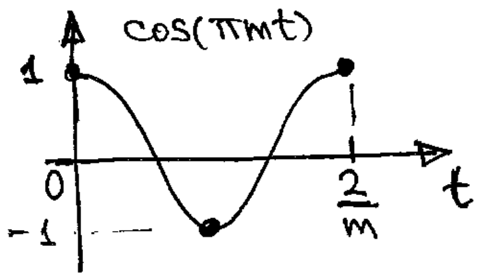
$$\frac{2\pi}{\pi m} = \frac{2}{m}$$

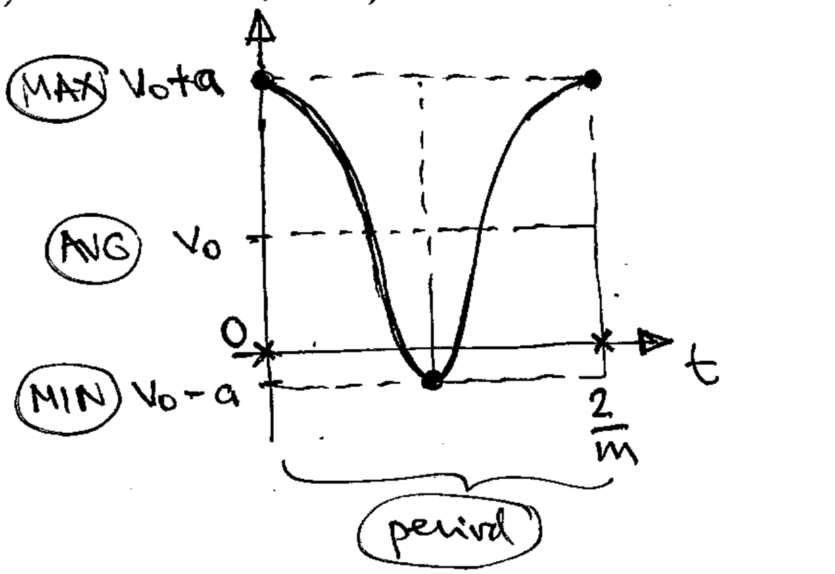
(b)[1] What is the amplitude of $\lambda_1(t)$?

a

(c)[3] Starting with the graph of $\cos t$ on the interval $[0, 2\pi]$, sketch the graph of $\lambda_1(t)$. (It is enough to show one period.)

$$\cos t \rightarrow \cos(\pi m t) \rightarrow a\cos(\pi m t) \rightarrow v_0 + a\cos(\pi m t)$$





(d)[2] Explain in words how to obtain the graph of $\lambda_2(t)$ from the graph of $\lambda_1(t)$.

$$\lambda_2(t) = v_0 + a\cos\left(\pi m\left(t + \frac{d}{\pi m}\right)\right)$$

shift <u>left</u> by $\frac{d}{\pi m}$

positive by assumption

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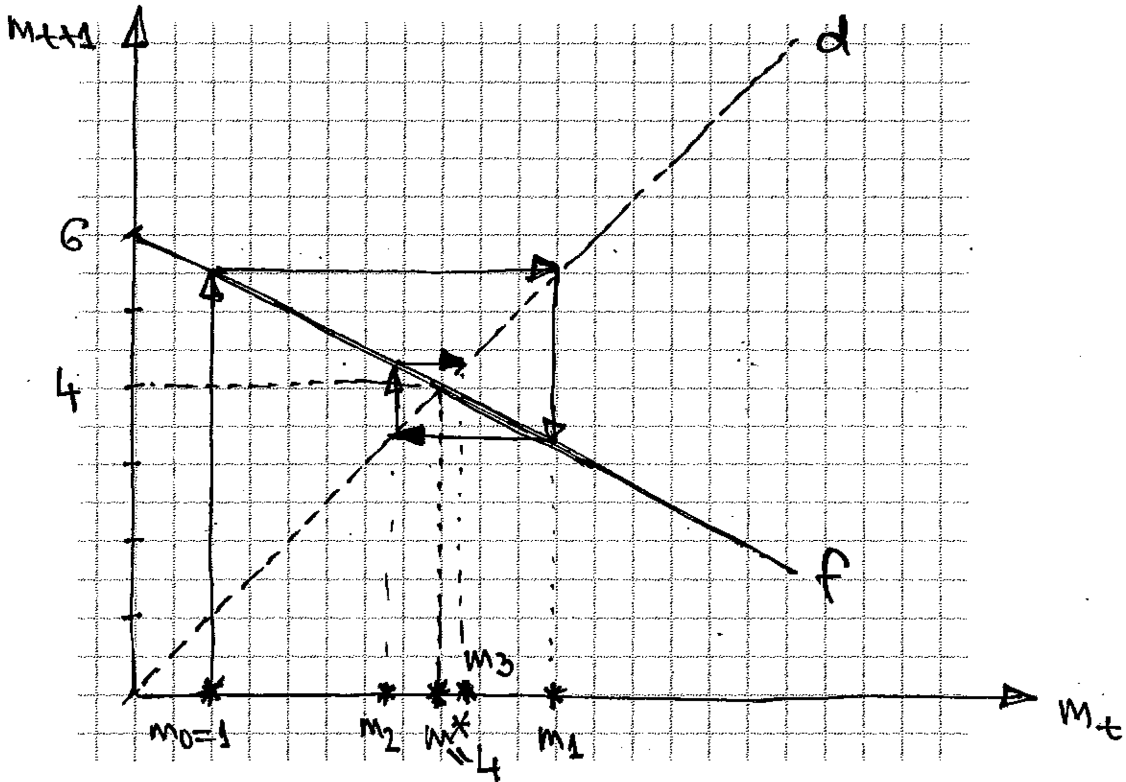
6. Consider the system $m_{t+1} = -0.5m_t + 6$, where $m_0 = 1$.
(a)[1] Find the equilibrium point(s) of the system.

$m^* = -0.5m^* + 6 \longrightarrow m^* = 4$

(b)[3] Starting with $m_0 = 1$, cobweb for three steps; i.e., in your diagram, show m_3. Also, indicate the equilibrium point(s) that you calculated in (a).



(c)[1] Calculate the value of m_3 algebraically and compare with your diagram in (b).

$m_0 = 1$

$m_1 = -0.5(1) + 6 = 5.5$

$m_2 = -0.5(5.5) + 6 = 3.25$

$m_3 = -0.5(3.25) + 6 = 4.375$

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7. It has been determined that the number of animal species, N, of certain length is inversely proportional to the cube root of their body length, L.

(a)[1] Express N as function of L.

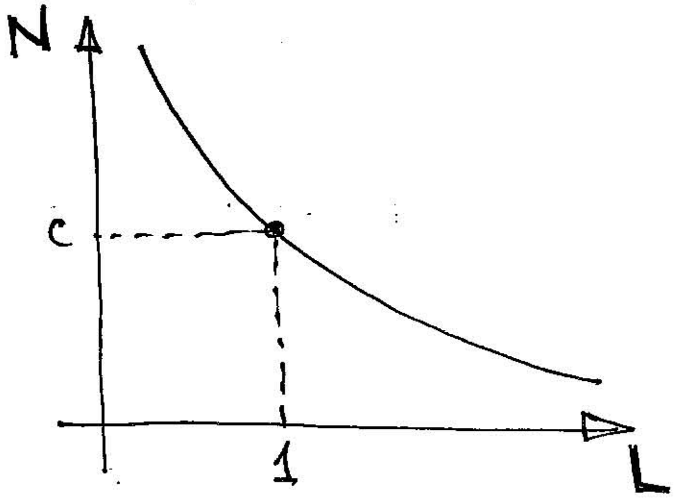
$$N(L) = N = c \cdot \frac{1}{\sqrt[3]{L}} \qquad c = \text{some real number}$$

(b)[2] By how much does N change if L triples?

$$N(3L) = c \cdot \frac{1}{\sqrt[3]{3L}} = \frac{1}{\sqrt[3]{3}} \cdot \underbrace{c \cdot \frac{1}{\sqrt[3]{L}}}_{\text{original } N(L)} \approx 0.693 \cdot N(L)$$

N changes by a factor of 0.693, ie, decreases to about 69.3%

(c)[2] Sketch the graph showing how N depends on L. Label the axes.



makes sense only when $L, N > 0$ (thus $c > 0$)

$$N = c \cdot L^{-1/3}$$

THE END