MATHEMATICS 1LS3 TEST 2

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Day Class
Duration of Examination: 60 minutes
McMaster University, 15 October 2013

FIRST NAME (please print): SOLUTIONS

FAMILY NAME (please print): ____________

Student No.: ____________

THIS TEST HAS 8 PAGES AND 7 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You must show work to receive full credit.

Problem	Points	Mark
1	6	
2	6	
3	6	
4	6	
5	7	
6	5	
7	4	
TOTAL	40	

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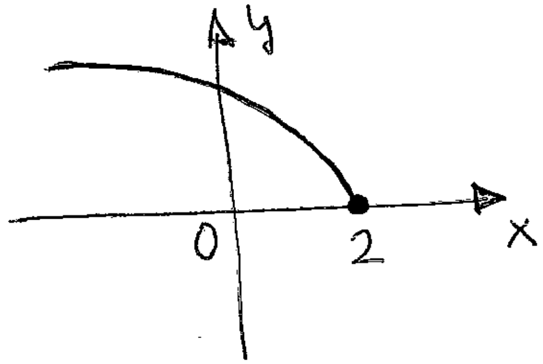
1. **Multiple choice questions: circle ONE answer. No justification is needed.**

(a)[3] Which of the following statements is/are true for the function $f(x) = \sqrt{2-x}$?

(I) $\lim_{x \to 2^+} f(x)$ does not exist (II) $\lim_{x \to 2^-} f(x)$ does not exist (III) $\lim_{x \to 1} f(x) = 1$

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

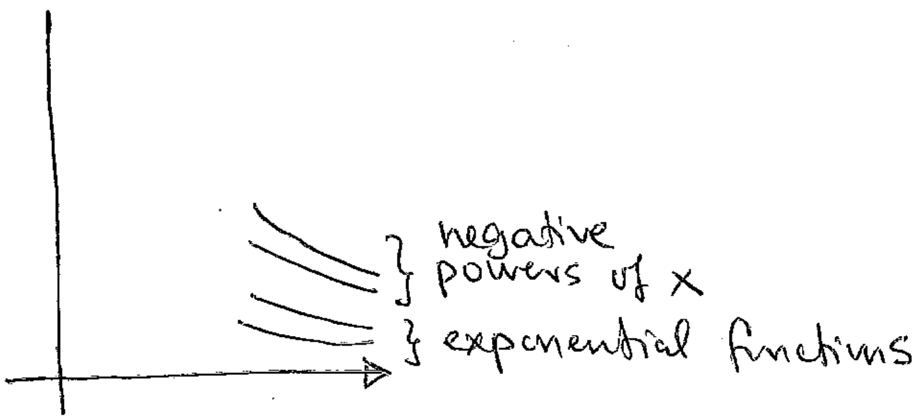
$\sqrt{2-x} = \sqrt{-x+2} = \sqrt{-(x-2)}$



(b)[3] Which of the functions approach(es) 0 more quickly than x^{-2} as $x \to \infty$?

(I) $f(x) = x^{-1.8}$ (II) $f(x) = x^{-2.5}$ (III) $f(x) = e^{-0.1x}$

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three



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2. Identify each statement as true or false (circle your choice). No justification is needed.

(a)[2] The line $y = \pi$ is a horizontal asymptote of the graph of $f(x) = \arctan(3x^2)$.

TRUE FALSE

$$\lim_{x\to\infty} \arctan(3x^2) = \arctan(\infty) = \frac{\pi}{2}$$

$$\lim_{x\to-\infty} \arctan(3x^2) = \arctan(\infty) = \frac{\pi}{2}$$

(b)[2] Cobwebbing the dynamical system $m_{t+1} = 0.97m_t$ starting at $m_0 = 10$ will approach the equilibrium value $m^* = 0$.

TRUE FALSE

$$m_t = 10 \cdot \underbrace{0.97^t}_{\text{approaches } 0 \text{ as } t\to\infty}$$

(c)[2] If $f(x) > 0$ for all $x > 0$, then $\lim_{x\to\infty} f(x) > 0$.

TRUE FALSE

$f(x) = \frac{1}{x} > 0$ for all $x > 0$

but $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x} = 0$

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Questions 3-7: You must show work to receive full credit.

3. Find each limit. If you claim that a limit does not exist, you need to be specific and explain why it is so (for instance, you have to show that the limit approaches infinity, or that the one-sided limits are not equal).

(a)[2] $\lim_{x\to 2^+} \frac{3x+1}{x-2} \overset{\oplus}{\underset{\oplus}{=}} \frac{7}{0} = +\infty$

(b)[2] $\lim_{x\to 4} \frac{\frac{1}{\sqrt{x}}-\frac{1}{2}}{x-4} = \frac{0}{0} = \lim_{x\to 4} \frac{\frac{2-\sqrt{x}}{2\sqrt{x}}}{x-4}$

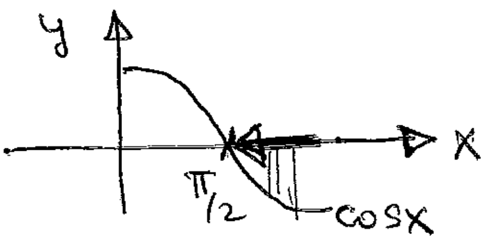
$$= \lim_{x\to 4} \frac{2-\sqrt{x}}{2\sqrt{x}(x-4)} \cdot \frac{2+\sqrt{x}}{2+\sqrt{x}}$$

$$= \lim_{x\to 4} \frac{\cancel{4-x}}{2\sqrt{x}\cancel{(x-4)}(2+\sqrt{x})} = -\lim_{x\to 4} \frac{1}{2\sqrt{x}(2+\sqrt{x})}$$

$$= -\frac{1}{16}$$

(c)[2] $\lim_{x\to(\pi/2)^+} x^2 \sec x$

$$= \lim_{x\to \frac{\pi}{2}^+} \frac{x^2}{\cos x} = \frac{\pi^2/4 \ \oplus}{0 \ \ominus} = -\infty$$



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4. An important quantity in the study of gating characteristics of ion channels is the channel opening probability $P_0 = \dfrac{1}{1+e^{\beta(G-A\tau)}}$, where τ is the membrane tension, and the parameters A, G, and β are positive.

(a)[3] Find the limit of P_0 as the membrane tension keeps increasing (i.e., approaches ∞)

$\tau \to \infty \to -A\tau \to -\infty$ (because $A > 0$)

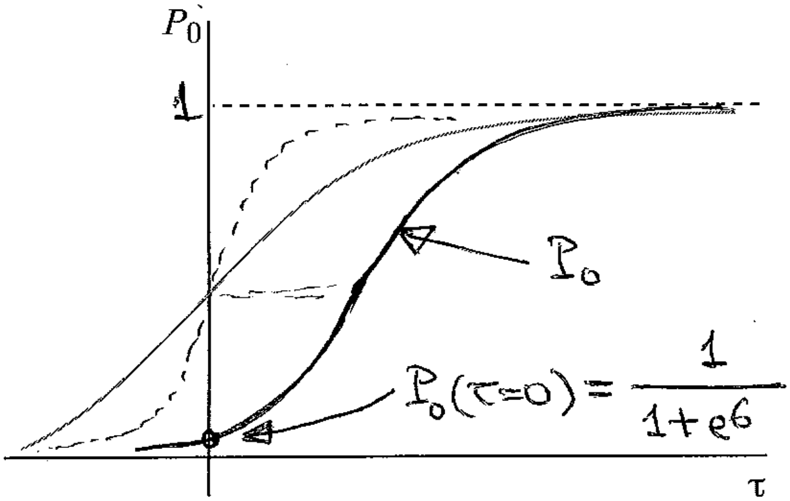
$\to G - A\tau = -A\tau + G \to -\infty$

$\to \beta(G-A\tau) \to -\infty$ (because $\beta > 0$)

so $$\lim_{\tau\to\infty} P_0 = \frac{1}{1+\underbrace{e^{-\infty}}_{0}} = 1$$

(b)[3] Drawn below is the graph of the function $y = \dfrac{1}{1+e^{-\tau}}$. In the same coordinate system, sketch the graph of the opening probability P_0 when $\beta = 1$, $G = 6$ and $A = 3$, for $\tau \geq 0$. Label the horizontal asymptote of P_0 and the vertical (P_0) intercept.

$P_0 = 1$ from (a)



$$P_0 = \frac{1}{1+e^{6-3\tau}} = \frac{1}{1+e^{-3(\tau-2)}}$$

$e^{-\tau} \longrightarrow e^{-3\tau} \longrightarrow e^{-3(\tau-2)}$

compress horizontally by a factor of 3 | shift right 2 units

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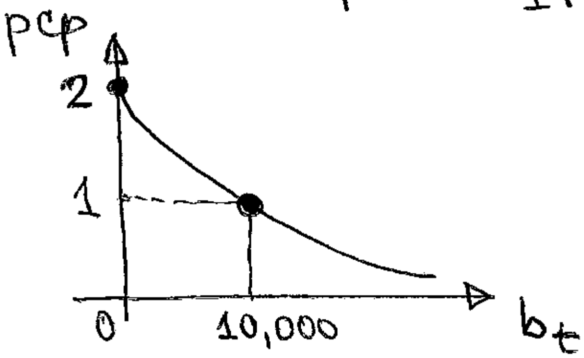
5. Consider the bacterial population model $b_{t+1} = \dfrac{2b_t}{1 + 0.0001 b_t}$, where b_t is the population count at time t.

(a)[2] Sketch the graph of the per capita production function. Label the coordinate axes and the vertical intercept, and identify coordinates of another point on the graph.

$$pcp = \frac{2}{1 + 0.0001 b_t}$$



(b)[2] Which feature of the graph in (a) makes this model a *limited* population model? Why?

decreasing: as the population increases, the number of new individuals per individual decreases toward 0 → which limits the growth size of the population

(c)[1] Check that $b^* = 10,000$ is an equilibrium.

when $b_t = 10,000$,

then $b_{t+1} = \dfrac{2(10,000)}{1+0.0001(10,000)} = \dfrac{20,000}{2} = 10,000$

(d)[2] You determined that $b^* = 10,000$ is a stable equilibrium. Explain how this fact justifies calling this model a *limited* population model.

if the population is above 10,000, it will move toward 10,000 (because of stability); i.e. it cannot grow arbitrarily large

likewise, if it is below 10,000, the population will increase toward 10,000

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6. The change in the size of a colony of bacteria in human navel can be modelled by $b_{t+1} = \left(Ae^{-mb_t} - De^{-2mb_t} + 1\right) b_t$, where $A, m, D > 0$.

(a)[3] Find all equilibria of the given dynamical system.

$$b^* = (Ae^{-mb^*} - De^{-2mb^*} + 1)\, b^*$$

$$b^*(Ae^{-mb^*} - De^{-2mb^*} + \cancel{1} - \cancel{1}) = 0 \rightarrow \underline{\underline{b^* = 0}}$$

$$Ae^{-mb^*} - De^{-2mb^*} = 0 \quad \Big| \cdot e^{2mb^*}$$

$$Ae^{mb^*} - D = 0$$

$$e^{mb^*} = \frac{D}{A} \rightarrow \underline{\underline{b^* = \frac{1}{m} \ln(D/A)}}$$

(c)[2] For which values of A and D does the given system have two different, biologically meaningful equilibria?

one equilibrium is $b^* = 0$ $\leftarrow$ will be different

so $b^* = \underbrace{\frac{1}{m}}_{\oplus} \ln(D/A) > 0$

$$\rightarrow \ln(D/A) > 0$$

$$D/A > e^0 = 1$$

ie, when $\underline{\underline{\frac{D}{A} > 1}}$ or $\underline{\underline{D > A}}$

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7. Consider the function

$$f(x) = \begin{cases} \dfrac{x^3 + 2x^2 - 8x}{x^2 - 4} & \text{if } x \neq 2 \\ 4 & \text{if } x = 2 \end{cases}$$

(a)[2] Is $f(x)$ continuous at $x = 2$? Explain why or why not.

USE DEFINITION

$$\lim_{x \to 2} f(x) \stackrel{?}{=} f(2) = 4$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x(x^2+2x-8)}{x^2-4}$$

$$= \lim_{x \to 2} \frac{x\cancel{(x-2)}(x+4)}{\cancel{(x-2)}(x+2)} = 3$$

not cont. at $x=2$

(b)[2] Is $f(x)$ continuous at $x = 0$? Explain why or why not.

near 0, $f(x)$ = rational function with denominator $\neq 0$

$\rightarrow$ continuous at $x=0$

THE END