MATHEMATICS 1LS3 TEST 3

Day Class
Duration of Examination: 60 minutes
McMaster University, 5 November 2013

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FIRST NAME (please print): SOLUTIONS

FAMILY NAME (please print): ______

Student No.: ______

THIS TEST HAS 8 PAGES AND 7 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You must show work to receive full credit.

Problem	Points	Mark
1	6	
2	6	
3	5	
4	6	
5	8	
6	3	
7	6	
TOTAL	40	

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Name:____________________
Student No.: ____________________

1. **Multiple choice questions: circle ONE answer. No justification is needed.**

(a)[3] It is known that a is in the domain of a function $f(x)$ and $f'(a) = 0$. Which statements is/are always true, i.e., hold for all functions $f(x)$?

(I) a is an inflection point of $f(x)$ NO

(II) $f(x)$ has a horizontal tangent at a ✓

(III) $f(x)$ has a relative extreme value (minimum or maximum) at a NO

(A) none (B) I only (C) II only (D) III only

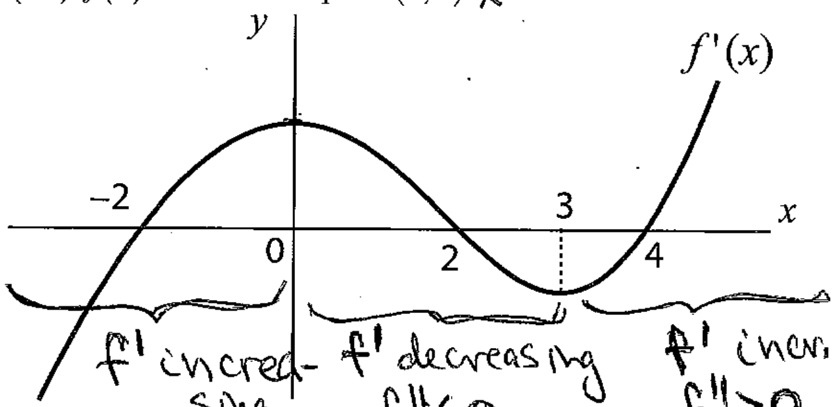
(E) I and II (F) I and III (G) II and III (H) all three

(b)[3] Given is the graph of the **derivative** $f'(x)$ of a function $f(x)$. Which of the following statements is/are true for the **function** $f(x)$?

(I) $x = 0$ is an inflection point of $f(x)$ ✓

(II) $f(x)$ is concave up on $(0, 3)$ X

(III) $f(x)$ is concave up on $(2, 4)$ X



f' increasing f'' > 0 f' decreasing f'' < 0 f' incre. f'' > 0

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

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Name: ____________________
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2. **Identify each statement as true or false (circle your choice). You do not need to justify your answer.**

(a)[2] The function $y = -1$ is the linear approximation of $f(x) = \sec x$ at $x = \pi$.

TRUE FALSE

$L_\pi(x) = f(\pi) + f'(\pi)(x - \pi)$

$\sec\pi = -1$ $f'(x) = \sec x \cdot \tan x$

$f'(\pi) = 0$

$L_\pi(x) = -1$

(b)[2] The function $f(x) = \arctan(x^3 + x)$ is increasing for all real ~~numbers~~ x.

TRUE FALSE

$$f'(x) = \frac{1}{1+(x^3+x)^2} \cdot (3x^2+1) > 0$$

(c)[2] If $x = 1$, then $\underbrace{\lim_{h \to 0} \frac{5^{x+h} - 5^x}{h}} = \ln 5$.

TRUE FALSE

this is the derivative of 5^x

$(5^x)' = 5^x \cdot \ln 5$

when $x = 1$, ... it is $\underline{5^1 \ln 5} = \underline{5 \cdot \ln 5}$

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Questions 3-7: You must show work to receive full credit.

3. The resistance R of the flow of blood through a blood vessel (assumed to have the shape of a cylindrical tube) is given by

$$R = \frac{K^{0.96}L(\gamma+1)^2}{d^4}$$

where L is the length of the tube, d is its diameter and $\gamma \geq 0$ is the curvature. The positive constant K represents the viscosity of the blood (viscosity is a measure of the resistance of fluid to stress; water has low viscosity, honey has high viscosity).

(a)[2] Find the derivative of R with respect to K and interpret your answer, i.e., explain what your answer implies for the dependence of R on the viscosity of the blood.

$$R' = \frac{L(\gamma+1)^2}{d^4} \cdot 0{,}96 \cdot K^{-0{,}04}$$

$$R' = \frac{0.96\, L(\gamma+1)^2}{d^4 K^{0{,}04}}$$

since $L, d, K > 0 \rightarrow R' > 0$.

ie, as the viscosity increases (blood becomes "thicker"), the resistance will increase

(b)[3] Find the derivative of R with respect to d and interpret your answer, i.e., explain what your answer implies for the dependence of R on the diameter of a blood vessel.

$$R' = K^{0{,}96} L(\gamma+1)^2 (-4)\, d^{-5}$$

$$= -\frac{4K^{0{,}96}L(\gamma+1)^2}{d^5}$$

$L, d, K > 0 \rightarrow R' < 0$

as the diameter increases (ie blood vessel becomes wider) the resistance will decrease

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4. (a)[1] State the assumption(s) of the Extreme Value Theorem.

$f(x)$ is continuous, defined on a closed interval $[a,b]$

(b)[1] State the conclusion(s) of the Extreme Value Theorem.

$f(x)$ has an absolute max. and an absolute min. on $[a,b]$

(c)[4] Find the absolute maximum and the absolute minimum of the function $f(x) = \dfrac{\ln x}{x^2}$ on the interval $[1, 2]$.

$$f'(x) = \frac{\frac{1}{x}\cdot x^2 - \ln x \cdot 2x}{x^4} = \frac{x - 2x\ln x}{x^4} = \frac{1-2\ln x}{x^3}$$

c.p.'s : $1 - 2\ln x = 0$, $\ln x = 1/2$, $x = e^{1/2} = \sqrt{e} \approx 1.65$

f' dne $\rightarrow x = 0$ (not a c.p. since not in domain of f)

x	$f(x)$
1	$\frac{\ln 1}{1^2} = 0$
2	$\frac{\ln 2}{4} \approx 0.173$
$e^{1/2}$	$\frac{\ln(e^{1/2})}{e} = \frac{1}{2e} \approx 0.184$

abs min. at $x = 1$

value: $f(1) = 0$

abs max at $x = e^{1/2}$

value: $f(e^{1/2}) = \frac{1}{2e} \approx 0.184$

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5. (a)[3] Find $f'(x)$, if $f(x) = 3^{\tan x} + (\tan x)^3 + (\tan 3)^3$.

$$f'(x) = 3^{\tan x} \cdot \ln 3 \cdot \sec^2 x$$
$$+ 3(\tan x)^2 \cdot \sec^2 x$$
$$+ 0$$

(b)[2] Find $f'(1)$ if $f(x) = \dfrac{a \ln x + b}{c \ln x + d}$.

$$f'(x) = \frac{a \cdot \frac{1}{x}(c \ln x + d) - (a \ln x + b) \cdot c \cdot \frac{1}{x}}{(c \ln x + d)^2}$$

$$f'(1) = \frac{ad - bc}{d^2}$$

(c)[3] Let $g(x) = x^2\sqrt{f(x)}$, where f is a differentiable function such that $f(1) = 4$ and $f'(1) = 1$. Find $g'(1)$.

$$g'(x) = 2x \cdot \sqrt{f(x)} + x^2 \cdot \frac{1}{2\sqrt{f(x)}} \cdot f'(x)$$

$$g'(1) = 2 \cdot \sqrt{\underbrace{f(1)}_{4}} + \frac{1}{2\sqrt{\underbrace{f(1)}_{4}}} \cdot \underbrace{f'(1)}_{1}$$
$$= 4 + \frac{1}{4} = \frac{17}{4}$$

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6. The following excerpt is taken from *Hybrid equation/agent-based model of ischemia-induced hyperemia and pressure ulcer formation predicts greater propensity to ulcerate in subjects with spinal cord injury.* Alexey Solovyev et al. PLoS Computational Biology. 9.5 (May 2013).

... explicit solution for $I_2(t)$ can be derived. This solution has the following form

$$I_2(t) = I_{\text{rest}}\left(1 + ae^{-p_1 t} + be^{-p_2 t}\right)$$

[3] Assume that $I_{\text{rest}} = 1$, $p_1 = 2$ and $p_2 = 3$. Find all critical points of $I_2(t)$. (Your answer will contain a and b.)

$$\underline{I_2(t) = 1 + ae^{-2t} + be^{-3t}}$$

$$I_2'(t) = -2ae^{-2t} - 3be^{-3t} = 0 \quad \Big| \cdot e^{3t}$$

$$-2ae^{t} - 3b = 0$$

$$e^{t} = -\frac{3b}{2a}$$

$$\underline{\underline{t = \ln\left(-\frac{3b}{2a}\right)}}$$

$I_2'(t)$ dne ... no such t

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7. (a)[4] Compute Taylor polynomials $T_2(x)$ and $T_3(x)$ for the function $f(x) = \sqrt[3]{x}$ near (or at) $a = 1$.

$f, f', \ldots$	at $a=1$
$f = x^{1/3}$	1
$f' = \frac{1}{3}x^{-2/3}$	$\frac{1}{3}$
$f'' = -\frac{2}{9}x^{-5/3}$	$-\frac{2}{9}$
$f''' = \frac{10}{27}x^{-8/3}$	$\frac{10}{27}$

$$T_3(x) = \overbrace{f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2}^{T_2(x)} + \frac{f'''(1)}{3!}(x-1)^3$$

$$T_3(x) = \underbrace{1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2}_{T_2(x)} + \frac{5}{81}(x-1)^3$$

(b)[2] Using the polynomial $T_2(x)$ from (a), find an estimate for $f(1.3) = \sqrt[3]{1.3}$.

$$T_2(x) = 1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2$$

$$\sqrt[3]{1{,}3} \approx T_2(1{,}3) = 1 + \frac{1}{3}(0{,}3) - \frac{1}{9}(0{,}3)^2$$

$$= 1 + 0{,}1 - 0{,}01 = \underline{\underline{1{,}09}}$$

(calculator value $\sqrt[3]{1{,}3} \approx 1{,}09139288$)

THE END