MATHEMATICS 1LS3 TEST 4

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Day Class
Duration of Examination: 60 minutes
McMaster University, 19 November 2013

FIRST NAME (please print): SOLUTIONS

FAMILY NAME (please print): ____________

Student No.: ____________

THIS TEST HAS 8 PAGES AND 7 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number. Any non-graphing calculator is allowed.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You must show work to receive full credit.

Problem	Points	Mark
1	6	
2	6	
3	5	
4	6	
5	6	
6	6	
7	5	
TOTAL	40	

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1. **Multiple choice questions: circle ONE answer. No justification is needed.**

(a)[3] Which of the following statements is/are true for a solution of the differential equation

$$P'(t) = 12P(t)\left(1 - \frac{450}{P(t)}\right)$$

(I) If $P(0) = 400$, then the solution $P(t)$ is an increasing function. $P' < 0$

(II) If $P(0) = 450$, then the solution $P(t)$ is an increasing function. $P' = 0$

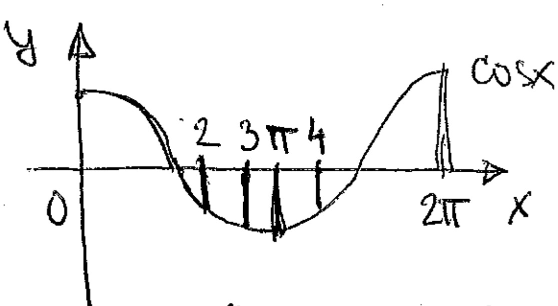
(III) If $P(0) = 500$, then the solution $P(t)$ is an increasing function. $P' > 0$

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

(b)[3] Which of these numbers is/are positive?

(I) $\int_0^2 \cos x\, dx$ (II) $\int_0^3 \cos x\, dx$ (III) $\int_0^4 \cos x\, dx$

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three



"area above" – "area below"

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2. Identify each statement as true or false (circle your choice). You do not need to justify your answer.

(a)[2] The function $f(x) = 2xe^{x^2}$ is an antiderivative of $g(x) = e^{x^2}$.

TRUE FALSE

$$(2xe^{x^2})' = 2e^{x^2} + 2xe^{x^2}(2x)$$

$$\neq e^{x^2}$$

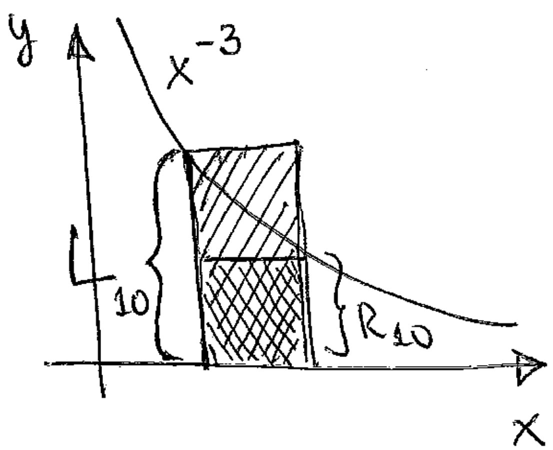
(b)[2] $\int_{-1}^{1} \frac{1}{x}\,dx = \ln|x|\Big|_{-1}^{1} = \ln|1| - \ln|-1| = 0.$

TRUE FALSE

ln|x| is not continuous on [-1,1]

(c)[2] The left and right Riemann sums of $f(x) = x^{-3}$ on $[1, 4]$ satisfy $L_{10} < R_{10}$.

TRUE FALSE



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Questions 3-7: You must show work to receive full credit.

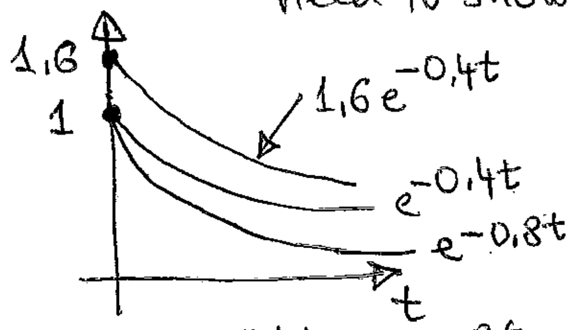
3. The rate of change of the amount of mercury deposited into Lake Huron from the surrounding land is given by

$$A'(t) = 1.6e^{-0.4t} - e^{-0.8t}$$

where t is time in hours and $A(0) = 1.6$. $A(t)$ is measured in kilograms. (The shoreline of Lake Huron, including all islands, is very long, measures over 6000 km.)

(a)[2] Explain why $A(t)$ is an increasing function.

need to show that $A'(t) > 0$ for $t \geq 0$



use graph

or:

$A'(t) = \underbrace{e^{-0{,}4t}}_{\oplus}\overset{\oplus}{(1{,}6 - e^{-0{,}4t})} > 0$

$(1{,}6 - e^{-0{,}4t})$ = 1,6 minus a number smaller or equal to 1

so $1{,}6e^{-0{,}4t} > e^{-0{,}8t}$

(b)[3] Find the amount of pollutant that leaks into Lake Huron in the first 24 hours.

$$\int_0^{24} (1{,}6e^{-0{,}4t} - e^{-0{,}8t})\,dt$$

$$= 1{,}6\frac{1}{-0{,}4}e^{-0{,}4t} - \frac{1}{-0{,}8}e^{-0{,}8t}\Big|_0^{24}$$

$$= -4e^{-0{,}4t} + 1{,}25e^{-0{,}8t}\Big|_0^{24}$$

$$= \underbrace{\left(-4e^{-0{,}4(24)} + 1{,}25e^{-0{,}8(24)}\right)}_{\approx -0{,}00027} - \underbrace{(-4 + 1{,}25)}_{= -2{,}75}$$

$\approx$ 2,75 kg

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4. Find the following indefinite and definite integrals.

(a)[2] $\int \frac{(2x-1)^2}{\sqrt[3]{x}}\,dx = \int \frac{4x^2-4x+1}{x^{1/3}}\,dx$

$$= \int \left(4x^{5/3} - 4x^{2/3} + x^{-1/3}\right) dx$$

$$= 4 \cdot \frac{x^{8/3}}{8/3} - 4 \cdot \frac{x^{5/3}}{5/3} + \frac{x^{2/3}}{2/3} + C$$

$$= \underline{\underline{\frac{3}{2} x^{8/3} - \frac{12}{5} x^{5/3} + \frac{3}{2} x^{2/3} + C}}$$

(b)[2] $\int_0^1 \frac{3}{1+4x^2}\,dx = 3 \int_0^1 \frac{1}{1+(2x)^2}\,dx$

$$= 3 \cdot \frac{1}{2} \arctan(2x)\Big|_0^1 = \frac{3}{2} \arctan(2x)\Big|_0^1$$

$$= \underline{\underline{\frac{3}{2} \arctan 2}} - \underbrace{\frac{3}{2} \arctan 0}_{0}$$

(c)[2] $\int \left(a\cos(\pi t) + b\sec^2 t\right)\,dt$

$$= \underline{\underline{a \cdot \frac{1}{\pi} \sin(\pi t) + b \tan t + C}}$$

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5. (a)[2] A sample of bacteria, initially at the temperature of 24.3^{O}C, is put into a -60^{O}C refrigerator. Let $T(t)$ be the temperature of the sample at time t. The temperature of the sample changes proportionally to the square of the difference between the temperature of the sample and the temperature of the refrigerator. Describe this event as an initial value problem (i.e., write down a differential equation and an initial condition). **Do not solve the equation.**

$$T'(t) = k \cdot (T(t) - (-60))^2$$

$$T(0) = 24.3$$

(b) [2] Describe the following event as initial value problem: an amoeba cell starts at a volume of 600 μm^3 and **loses** volume at the rate of $1.4e^{0.02t}$ μm$^3/s$.

$$V(0) = 600$$

$$V'(t) = -1{,}4\, e^{0{,}02t}$$

(c)[2] Find the solution of the problem in (b).

$$V(t) = \int (-1{,}4\, e^{0.02t})\, dt$$

$$= -1{,}4 \cdot \frac{1}{0{,}02} e^{0.02t} + C$$

$$= -70\, e^{0{,}02t} + C$$

$$V(0) = 600 \rightarrow 600 = -70 + C$$

$$C = 670$$

$$V(t) = -70\, e^{0{,}02t} + 670$$

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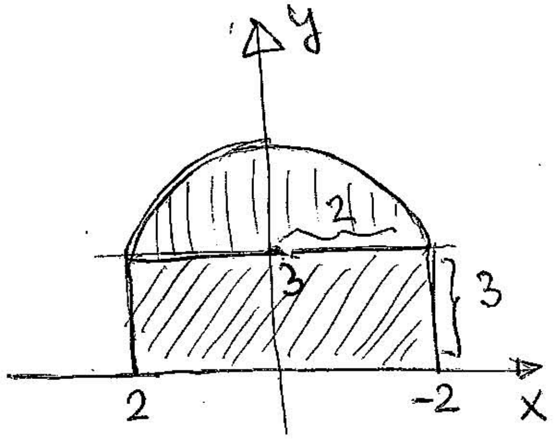
6. (a)[3] Compute $\int_{-2}^{2}\left(3+\sqrt{4-x^2}\right)dx$ by interpreting the definite integral as area.



" rectangle + semicircle

$= \underline{\underline{12 + 2\pi}}$

(b)[3] Using L'Hôpital's rule, calculate $\lim_{x\to 0}\frac{e^{x^2}-1-x^2}{x^4} = \frac{0}{0}$

$\overset{LH}{=} \lim_{x\to 0}\frac{e^{x^2}\cdot 2x - 2x}{4x^3} = \lim_{x\to 0}\frac{2e^{x^2}-2}{4x^2} = \lim_{x\to 0}\frac{e^{x^2}-1}{2x^2}$

simplify   simplify

$\overset{LH}{=} \lim_{x\to 0}\frac{e^{x^2}\cdot \cancel{2x}}{4\cancel{x}} = \underline{\underline{\frac{1}{2}}}$

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7. Consider the initial value problem $y' = (3t-2)$, $y(0) = 2$.

(a)[2] Find an approximation of $y(0.5)$ using two steps of Euler's method with the step size $\Delta t = 0.25$.

$$\left.\begin{array}{l} t=0 \\ y(0)=2 \end{array}\right\} \to L(t) = \underbrace{y(0)}_{2} + \underbrace{y'(0)}_{-2}(t-0)$$

$$L(t) = 2-2t$$

one step $\left\{ \begin{array}{l} t=0{,}25 \\ y(0{,}25) \approx L(0{,}25) = \underline{\underline{1.5}} \end{array}\right.$

$$L(t) = \underbrace{y(0{,}25)}_{1{,}5} + \underbrace{y'(0{,}25)}_{-1{,}25}(t-0{,}25)$$

step two $\left\{ \begin{array}{l} t=0{,}5 \\ y(0{,}5) \approx L(0{,}5) = 1{,}5 - 1{,}25(0{,}25) = \underline{\underline{1{,}1875}} \end{array}\right.$

(b)[2] Using antidifferentiation, find the exact solution of the given initial value problem.

$$y = \int (3t-2)\,dt = 3\frac{t^2}{2} - 2t + C$$

$$y(0) = 2 \to C = 2$$

$$\boxed{y(t) = \frac{3t^2}{2} - 2t + 2}$$

(c)[1] Using (b), find the true value of $y(0.5)$ (thus checking your approximation in (a)).

$$y(0{,}5) = \frac{3(0{,}25)}{2} - 2(0{,}5) + 2 = \underline{1{,}375}$$

THE END