1A03 – Calculus I for Science
(Section C02)

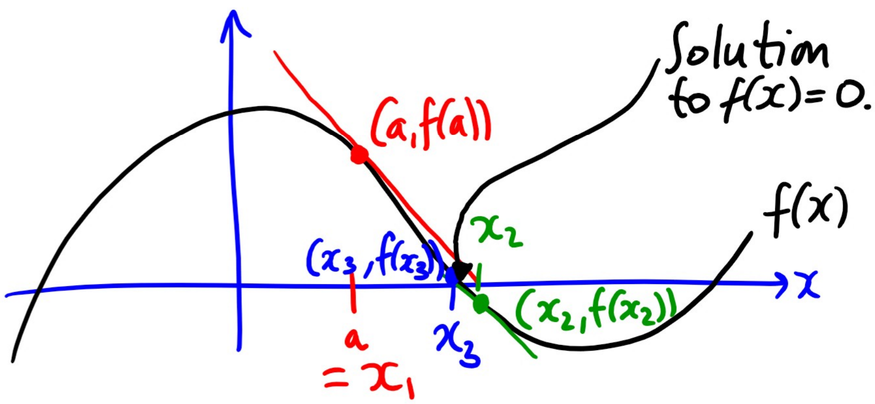
Lecture 10

Last time

Newton's Method

Use tangent lines to $y=f(x)$ to "home in on" solutions to

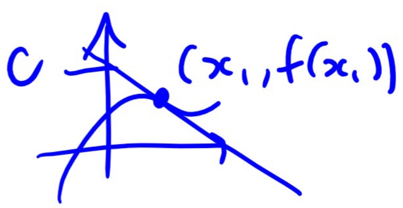
$$f(x)=0.$$



More precisely: Set $x_1 = a$

Then x_2 is x-interecept of tangent line to $(a, f(a)) = (x_1, f(x_1))$

Find x_2: Tangent line equation



$$y = f'(x_1)x + c$$

$$c = f(x_1) - f'(x_1)x_1$$

OR rearranging $y = f'(x_1)(x-x_1)+f(x_1)$

So $0 = f'(x_1)(x_2 - x_1) + f(x_1)$

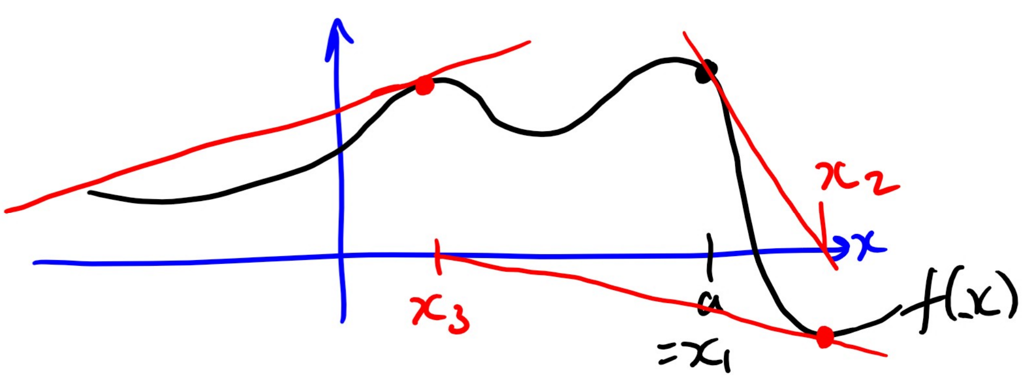
Solve for $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}.$

Iterate: $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$.

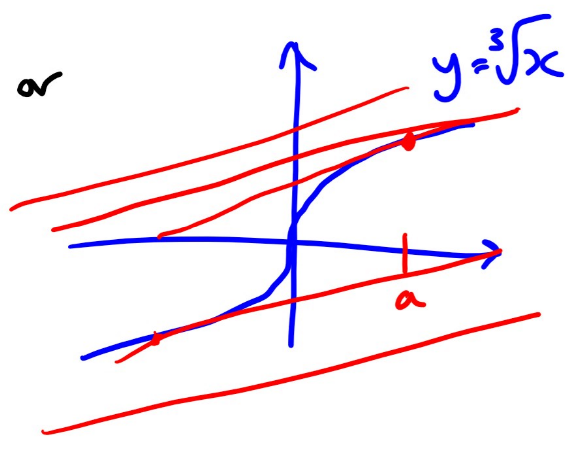
In general, $\boxed{x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}}$. For $n \geq 1$.

We keep going until successive steps agree to a usually large, usually predetermined # of decimal places.

Remarks 1. For some curves / initial guesses, this method does not work.



or



2. In the case that this works but there is more than one solution our initial guess $(a = x_1)$ dictates which solution we end up approximating.

Example Approximate a root of the equation
$$x^3 = x^2 + 2x - 1$$ to 2 decimal places

Solution First set it up as an equation of the form $f(x) = 0$.

$$\underbrace{x^3 - x^2 - 2x + 1}_{f(x)} = 0.$$

Now make an initial guess. How?

- sketch graph
- try a few values e.g. whole numbers & choose one with output "close" to 0

e.g. Here: Try $x = 1$: $f(1) = 1 - 1 - 2 + 1 = -1$

Try $x = 2$: $f(2) = 8 - 4 - 4 + 1 = 1$

So let us set $x_1 = 1$. (Both 1 & 2 would be OK choices.)

Then $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 1 - \dfrac{(-1)}{3-2-2}$

Need to work out $f'(x) = 3x^2 - 2x - 2$

$= 1 - \left(\dfrac{-1}{-1}\right)$

$= 0$ ← be careful! Not a solution. New x-value

Then $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

$$= 0 - \frac{f(0)}{f'(0)} = -\frac{1}{-2} = \frac{1}{2} = 0.5$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = \frac{1}{2} - \frac{f(1/2)}{f'(1/2)} = \ldots = \frac{4}{9}$$

$$= \underline{\underline{0.444\ldots}}$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = \frac{4}{9} - \frac{f(4/9)}{f'(4/9)} = \ldots = \frac{4}{9} + \frac{1}{1674}$$

$$= \underline{\underline{0.44}}504\ldots$$

(In fact, exact solution is 0.445042...)

<u>Example</u> Approximate $\sqrt[7]{1000}$ to 8 decimal places using Newton's Method.

<u>Solution</u> Find $f(x)$ so that a solution to $f(x)=0$ is $\sqrt[7]{1000}$.

Write $x = \sqrt[7]{1000} \rightarrow$ so $x^7 = 1000$

(Cannot use $x - \sqrt[7]{1000} = 0$ as our $f(x)=0$, as this will require plugging into our calculations the value of $\sqrt[7]{1000}$, which is exactly what we are trying to approximate!)

and $\underbrace{x^7 - 1000}_{f(x)} = 0$

Need initial guess

$f(1) = -999$

$f(2) = -872$

$f(3) = 1187 \longrightarrow$ We'll use $3 = a = x_1$

Either of these is as good as the other; both around 1000 away (which is "close" for this version of the problem).

Now $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3 - \frac{1187}{\cancel{7 \cdot 3^6} \; 5103} = 2.76739.$

Work out $f'(x)$

$= 7x^6$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.76\ldots - \frac{243.066\ldots}{3144.284\ldots}$$

$$= 2.690087\ldots$$

$$x_4 = \ldots = 2.682756\ldots$$

$$x_5 = \ldots = 2.68269579|9$$

$$x_6 = \ldots = \underline{2.68269579}|5$$

3.4 Chain Rule

Next time.