1A03 – Calculus I for Science
(Section C02) Lecture 11

Today Chain Rule

If $f(x) = (g \circ h)(x) = g(h(x))$ and $h(x)$ is differentiable at x and $g(x)$ is differentiable at $h(x)$ then $f(x)$ is differentiable at x and $f'(x) = g'(h(x)) \cdot h'(x)$.

Alternatively if $y = f(x) = g(u)$ where $u = h(x)$

then $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$ ← NOT quotients; but pretend that they are & imagining cancelling out.

Example Let $y = e^{x^2}$. Find $\dfrac{dy}{dx}$.

inner: x^2 outer: e

Solution Say $y = e^u$, $u = x^2$

$$\frac{dy}{du} = e^u \qquad \frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot 2x = \underline{\underline{2xe^{x^2}}}.$$

In general $\dfrac{d}{dx}\left(e^{h(x)}\right) = h'(x)e^{h(x)}$.

Example Let $f(x) = \underbrace{\tan}_{\text{outer}}(\underbrace{5x-1}_{\text{inner}})$. Find $f'(x)$.

Solution Let $g(u) = \tan(u)$
$h(x) = 5x - 1$

. So $f'(x)$
$= g'(h(x)).h'(x)$
$= \sec^2(h(x)).h'(x)$
$= 5\sec^2(5x-1)$.

Example Let $y = \sin(e^{2x})$. Find $\frac{dy}{dx}$.

Solution Let $y = \sin(u)$
$u = e^{2x}$ $\longrightarrow$ $u = e^v$
$v = 2x$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$
$$= \cos(u).(2e^{2x})$$
$$= \cos(e^{2x}).(2e^{2x})$$
$$= 2e^{2x}\cos(e^{2x})$$

Using Chain Rule:
$$\frac{du}{dx} = \frac{du}{dv}\cdot\frac{dv}{dx}$$
$$= e^v . 2$$
$$= 2e^{2x}$$

In general if $y = f(x)$
$= g(h(k(x)))$

we have $f'(x) = g'(h(k(x)))\, h'(k(x))\, k'(x)$

If we set $y = g(u)$
$u = h(v)$
$v = k(x)$, then $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$.

<u>Example</u> Let $f(x) = b^x$, $b > 0$, $b \neq 1$.
What is $f'(x)$?

<u>Solution</u> We already saw that $f'(x) = f'(0)\, b^x$.

Now we can be cleverer:

$$\begin{aligned} f(x) &= b^x \\ &= e^{\ln(b^x)} \\ &= e^{x(\ln b)} \end{aligned}$$

So

$$\begin{aligned} f'(x) &= (x(\ln b))' \, e^{x(\ln b)} \\ &= (\ln b)\underbrace{e^{x(\ln b)}}_{b^x} \\ &= (\ln b)\, b^x. \end{aligned}$$

<u>Example</u> If $h(x)$ is differentiable and n is a real #, find $\frac{d}{dx}\left(h(x)^n\right)$.

$\left(\begin{array}{l} (5x^2+1)^{100} \ ?? \\ (5x^2+1)^{-1.7} \ ????? \end{array} \right)$

<u>Solution</u>

$g(u) = u^n$
$u = h(x)$

So

$$\begin{aligned} \frac{d}{dx}\left(h(x)^n\right) &= g'(\overset{h(x)}{u})\, h'(x) \\ &= \underline{n(h(x))^{n-1} h'(x)}. \end{aligned}$$

(Chain Rule + Power Rule)

<u>Example</u> If $f(x) = g(h(x))$, find $f'(-1)$ if $h(-1) = 2$, $h'(-1) = 3$, $g'(2) = 4$.

Solution $f'(-1) = g'(h(-1)) \cdot h'(-1)$ (Chain Rule at $x=-1$)

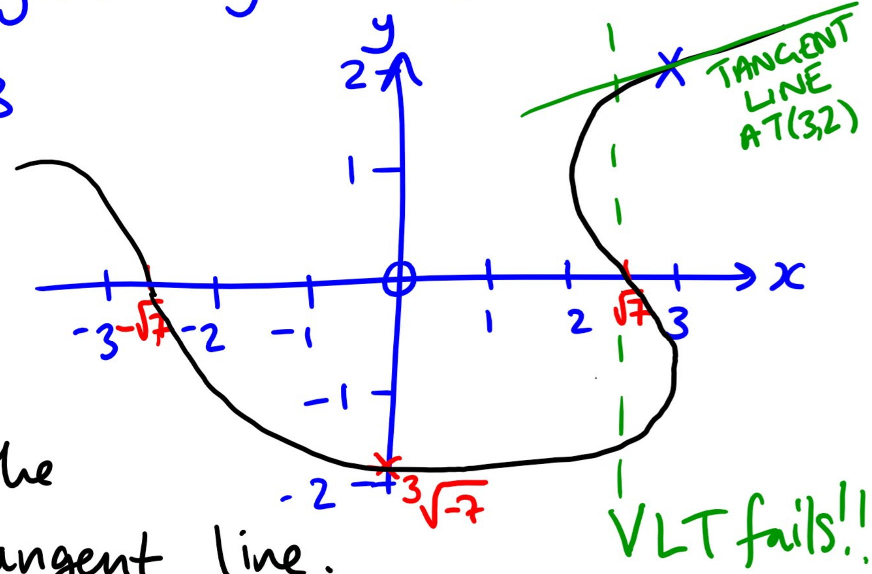
$$= g'(2) \cdot 3$$

$$= 4 \cdot 3 = 12.$$

3.5 Implicit Differentiation

Consider the curve given by the equation

$$x^2 + xy = 7 + y^3$$



This is NOT the graph of a function.

But at each point of the curve there IS a tangent line.

Problem Find the equation of the tangent line to the curve at the point $(3,2)$

$$3^2 + 3 \cdot 2 = 15 = 7 + 2^3$$

Solution One idea: solve for y in terms of x & solve explicitly

$\rightarrow$ essentially impossible here.

Instead: Use implicit differentiation.

Idea: The equation we have $(x^2 + xy = 7 + y^3)$ defines y implicitly in terms of several functions of x.

So: Differentiate both sides of the equation (with respect to $\underline{\underline{\underline{x}}}$) and solve for y'.

(Think: $y=f(x)$)

$$x^2 + xy = 7 + y^3$$

$$\rightarrow (x^2 + xy)' = (7 + y^3)'$$

$$2x + x'y + xy' = (y^3)'$$

$$2x + y + xy' = \underline{\underline{(y^3)'}}$$

TBC ...

Hint: Use the Chain Rule + Power Rule !!!