1A03 – CALCULUS I FOR SCIENCE

(SECTION C02) Lecture 12

Last time

IMPLICIT DIFFERENTIATION

We can differentiate equations & find tangents to curves that are not necessarily graphs of functions

e.g. (Example in progress)

$$x^2 + xy = 7 + y^3$$

$$\Rightarrow (x^2+xy)' = (7+y^3)'$$

$'$: w.r.t. x

$$\Rightarrow 2x + y + xy' = (y^3)' \ldots$$

$$= 3y^2 \cdot y' \quad \text{(Power Rule + Chain Rule)}$$

We want slope of tangent at point $(x,y) = (3,2)$ so solve for y' at $(3,2)$:

$$y'(x - 3y^2) = -2x - y$$

$$y' = \frac{-(2x+y)}{x-3y^2}$$

So at $(3,2)$, $y' = \dfrac{-(6+2)}{3-12} = \dfrac{8}{9}$.

Equation of tangent line at $(x,y) = (3,2)$:

$$y = y'(x-3) + 2 = \frac{8}{9}x - \frac{2}{3}.$$

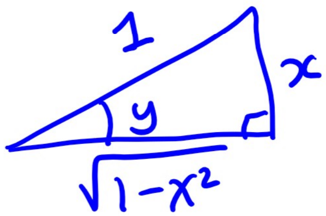
Inverse Trig. Functions

Let $y = \arcsin(x)$

Find $\frac{dy}{dx}$.

Well $\sin(y) = x$

Implicit differentiation: $\cos(y) . y' = 1$



$$y' = \frac{1}{\cos(y)}.$$

$$= \frac{1}{\cos(\arcsin(x))}$$

$$y' = \frac{1}{\sqrt{1-x^2}}.$$

Similarly $\frac{d}{dx}(\arccos(x)) = -\frac{1}{\sqrt{1-x^2}}.$

$$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$$

In general: Derivatives of Inverse Functions

Suppose $f(x)$ is 1-1, differentiable, and suppose f^{-1} is also differentiable.

What is $(f^{-1})'$?

Well if $y = f^{-1}(x)$, then $f(y) = x$.

Differentiate implicitly: $f'(y) . y' = 1$

$$y' = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

So $\boxed{\left(f^{-1}(x)\right)' = \dfrac{1}{f'(f^{-1}(x))}}$.

<u>Example</u> Let $f(x) = \sqrt{x} + x^2 + 1$, $x \geq 0$

Find the slope of the tangent line to $y = f^{-1}(x)$ at the point $(3, 1)$

Notice: $f(1) = 3$, so $f^{-1}(3) = 1$.
So $(3,1)$ is a point on the graph of f^{-1}!

<u>Solution</u> $\left.\dfrac{dy}{dx}\right|_{x=3} = \left.\dfrac{1}{f'(f^{-1}(x))}\right|_{x=3} = \dfrac{1}{f'(f^{-1}(3))} = \dfrac{1}{f'(1)}$

Evaluate at $x = 3$

? What is? $f'(x) = \frac{1}{2}x^{-1/2} + 2x$

$f'(1) = \frac{1}{2} + 2 = \frac{5}{2}$ plug in above!

So answer (slope) is $\dfrac{1}{f'(1)} = \dfrac{2}{5}$

$(f^{-1}(3))' =$

3.6 Derivatives of log. functions

For $y = \log_b x$, $b > 0$, $b \neq 1$,

we have $b^y = x$ so by implicit differentiation

we have $y' = \dfrac{1}{(\ln b)\, b^y} = \dfrac{1}{(\ln b)\, x}$.

So if $b = e$ and $\boxed{f(x) = \ln x}$, we get

$$\boxed{f'(x) = \frac{1}{x}}.$$

<u>Example</u> Let $f(x) = \ln(\sin^2 x)$. Find $f'(x)$.

<u>Solution</u>

$$y = f(x)$$

$$y = g(u) = \ln(u) \qquad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$u = h(v) = v^2$$

$$v = k(x) = \sin x$$

So
$$f'(x) = \frac{1}{u} \cdot 2v \cdot \cos x = \frac{1}{\sin^{\cancel{2}} x} \cdot 2\cancel{\sin x} \cdot \cos x$$

$$= \underline{\underline{2\cot(x)}}.$$

In general, by Chain Rule:

$$\frac{d}{dx}\left(\ln(h(x))\right) = \frac{1}{h(x)} \cdot h'(x)$$

<u>Logarithmic Differentiation</u>

$y = f(x)$, f hard to differentiate.

We "take logs" (ln applied to both sides):

$$\ln y = \ln f(x)$$

$$\frac{y'}{y} = \frac{d}{dx}(\ln f(x)) \longrightarrow$$

The idea is that $\ln(f(x))$ should be much easier to differentiate than $f(x)$.

Example Let $y = \sqrt{\dfrac{x+1}{x^3-1}}$. Find y'.

Solution

$$y = \frac{(x+1)^{1/2}}{(x^3-1)^{1/2}}$$

Could use Quotient Rule ...

Or : take logs : $\ln y = \ln\left(\dfrac{x+1}{x^3-1}\right)^{1/2}$

$$\Rightarrow \ln y = \frac{1}{2}\ln\left(\frac{x+1}{x^3-1}\right)$$

$$\Rightarrow \ln y = \frac{1}{2}\left(\ln(x+1) - \ln(x^3-1)\right).$$

Implicit derivatives:

(i.e. differentiate both sides of the equation w.r.t. x.)

$$\frac{y'}{y} = \frac{1}{2}\left(\frac{1}{x+1} - \frac{3x^2}{x^3-1}\right)$$

$$\Rightarrow y' = \frac{1}{2}\left(\frac{1}{x+1} - \frac{3x^2}{x^3-1}\right)\cdot \underbrace{\left(\frac{x+1}{x^3-1}\right)^{1/2}}$$

substitute in y as a function of x from above

Simplify

$$= \frac{1}{2}\left(\frac{x^3-1-3x^2(x+1)}{(x+1)(x^3-1)}\right)\frac{(x+1)^{1/2}}{(x^3-1)^{1/2}}$$

$$= \frac{1}{2}\left(\frac{-2x^3-3x^2-1}{(x+1)^{1/2}(x^3-1)^{3/2}}\right).$$

$$= -\frac{1}{2}\left(\frac{2x^3+3x^2+1}{(x+1)^{1/2}(x^3-1)^{3/2}}\right).$$