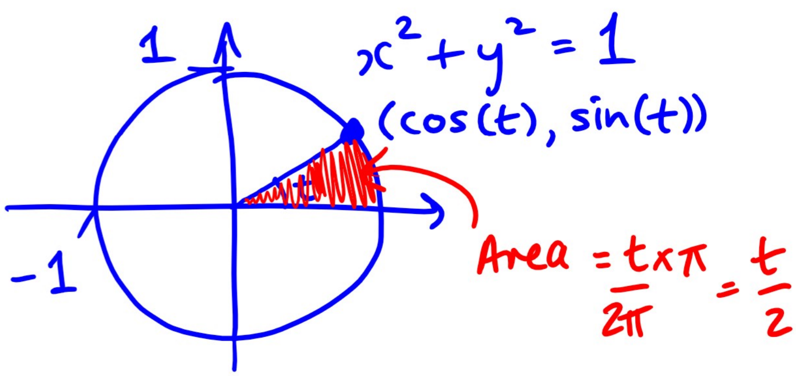
1A03 – Calculus I for Science (Section C02)

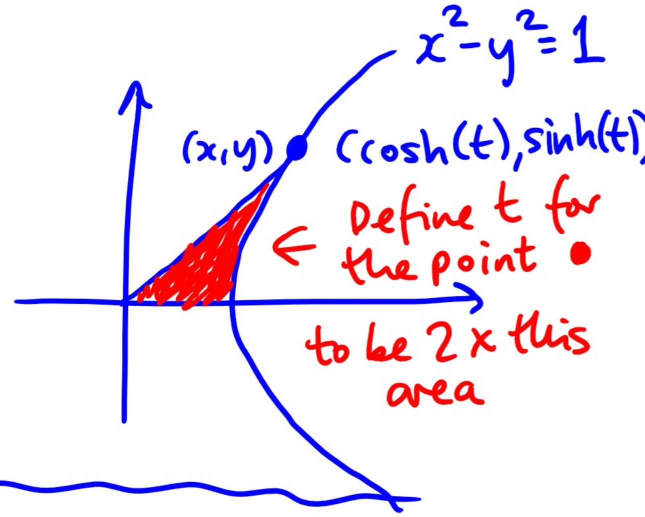
Lecture 13

Today — Hyperbolic Functions

Family of functions related to trig. functions & exponential functions.

"Hyperbolic functions are to hyperbolas as trig. functions are to circles"





Definition

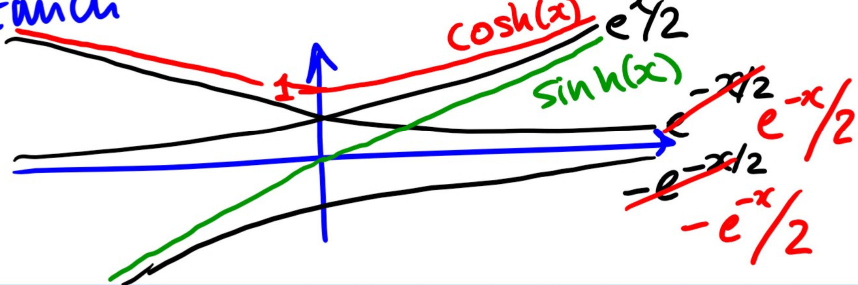
"shine"
"sinch"

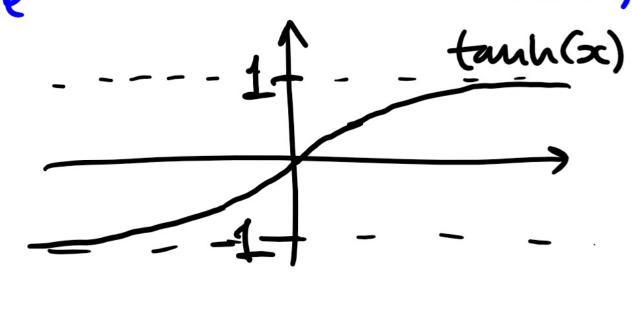
$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth(x) = \frac{1}{\tanh(x)}$$

"tanch"





Identities

$\sinh(-x) = -\sinh(x)$ (odd)

$\cosh(x) = \cosh(-x)$ (even)

$\tanh(-x) = -\tanh(x)$ (odd)

$$\cosh^2(x) = \frac{(e^x + e^{-x})^2}{2^2} = \frac{e^{2x} + 2 + e^{-2x}}{4}$$

$$\sinh^2(x) = \frac{(e^x - e^{-x})^2}{2^2} = \frac{e^{2x} - 2 + e^{-2x}}{4}$$

$$\cosh^2(x) + \sinh^2(x) = \frac{\cancel{2}e^{2x} + \cancel{2}e^{-2x}}{\cancel{4}\,2} = \cosh(2x)$$

$$\underbrace{\cosh^2(x) - \sinh^2(x)}_{} = \frac{2-(-2)}{4} = \underline{1}$$

Divide by $\cosh^2(x)$: $1 - \tanh^2(x) = \operatorname{sech}^2(x)$

or by $\sinh^2(x)$: $\coth^2(x) - 1 = \operatorname{csch}^2(x)$.

The above is a special case of the below with $y = x$ below.

In general: $\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\cdot\sinh(y)$

$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$.

Exercises

Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh(x)) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x \cancel{-}^{+} e^{-x}}{2} = \cosh(x)$$

$$\frac{d}{dx}(\cosh(x)) = \frac{d}{dx}\left(\frac{e^x+e^{-x}}{2}\right) = \frac{e^x-e^{-x}}{2} = \sinh(x)$$

$$\frac{d}{dx}(\tanh(x)) = \frac{d}{dx}\left(\frac{\sinh(x)}{\cosh(x)}\right) = \frac{\cosh^2(x)-\sinh^2(x)}{\cosh^2(x)}$$

$$= \frac{1}{\cosh^2(x)} = \operatorname{sech}^2(x).$$

Example Let $f(x) = \ln(\sinh(x))$. Find $f'(x)$.

Solution $$f'(x) = \frac{(\sinh(x))'}{\sinh(x)} = \frac{\cosh(x)}{\sinh(x)} = \coth(x).$$

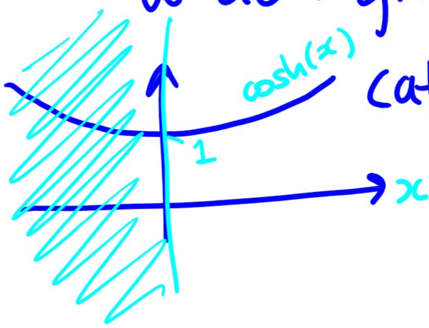
Inverse Hyperbolic Functions

Which graphs pass the HLT?

$\sinh(x)$ ✓

$\tanh(x)$ ✓

↳ here $-1<y<1$



catenary

$\cosh(x)$ $x \geq 0$ ✓

↳ and here $y \geq 1$.

$$y = \sinh(x) = \frac{e^x - e^{-x}}{2} \rightarrow 2y = e^x - \frac{1}{e^x}$$

$$(2y)e^x = (e^x)^2 - 1$$

$$(e^x)^2 - (2y)e^x - 1 = 0$$

Quadratic formula: $$e^x = \frac{\cancel{2}y \pm \sqrt{\cancel{4}y^2 + \cancel{4}1}}{\cancel{2}}$$

$\leftarrow \sqrt{4y^2+4}$
$= \sqrt{4(y^2+1)}$
$= \sqrt{4}\sqrt{y^2+1}$
$= 2\sqrt{y^2+1}$

$$e^x = y \pm \sqrt{y^2+1}.$$

$e^x > 0$ so only valid solution is $e^x = y + \sqrt{y^2+1}$

$$x = \ln\left(y + \sqrt{y^2+1}\right)$$

reverse roles of x & y

$$\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right).$$

Possible y-values of cosh are ≥ 1

Similarly
$$\cosh^{-1}(x) = \ln\left(x - \sqrt{x^2-1}\right), \quad x \geq 1$$
$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad -1 < x < 1$$

Possible y-values of tanh are between -1 and 1.

Derivatives of Inverse Hyp. Functions

Can differentiate above formulas or use our $\left(f^{-1}(x)\right)' = \dfrac{1}{f'(f^{-1}(x))}$.

e.g. $\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)$

$$\left(\sinh^{-1}(x)\right)' = \frac{\left(x + \sqrt{x^2+1}\right)'}{x + \sqrt{x^2+1}} \quad \cdots\cdots \quad (x^2+1)^{1/2}$$

$$= \frac{1 + \cancel{\tfrac{1}{2}} \cdot \cancel{2}x \cdot (x^2+1)^{-1/2}}{x + \sqrt{x^2+1}} \quad \to \frac{1}{\sqrt{x^2+1}}$$

Multiply through by $\sqrt{x^2+1}$.

$$= \frac{\cancel{\sqrt{x^2+1} + x}}{\sqrt{x^2+1}\,\cancel{\left(x + \sqrt{x^2+1}\right)}} = \frac{1}{\sqrt{x^2+1}}.$$

Also $\left(\cosh^{-1}(x)\right)' = \dfrac{1}{\sqrt{x^2-1}}$ $\quad \left(\tanh^{-1}(x)\right)' = \dfrac{1}{1-x^2}.$