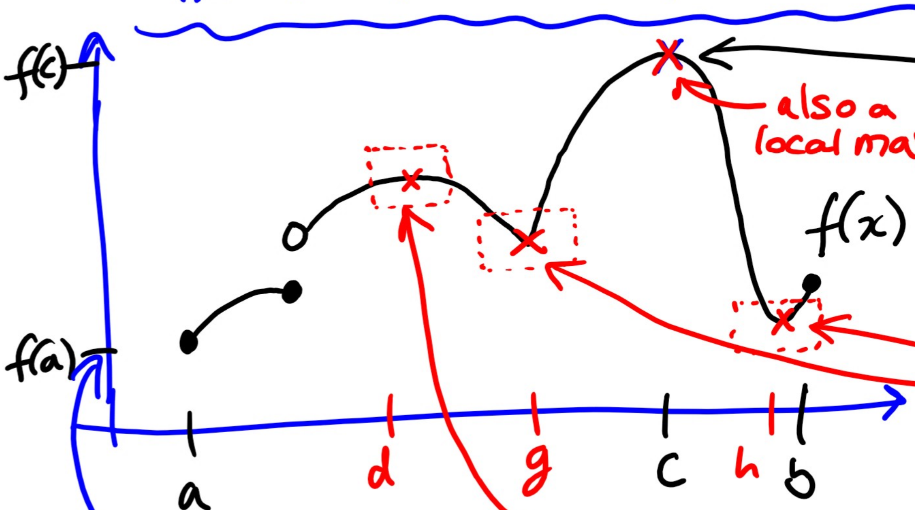
1A03 – CALCULUS I FOR SCIENCE

(SECTION C02) Lecture 14

Today + beyond: USING DERIVATIVES TO SOLVE PROBLEMS

4.1 Maximum & Minimum Values



largest of $f(x)$ on $[a,b]$ = absolute maximum of $f(x)$ on $[a,b]$ i.e. $f(x) \leq f(c)$ for all x in $[a,b]$

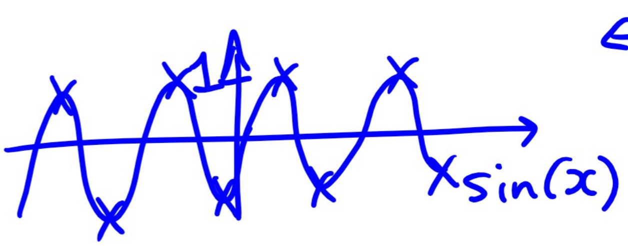
local minima of $f(x)$: $f(x) \geq f(g)$ for all x near g; $f(x) \geq f(h)$ for all x near h

local maximum of $f(x)$: $f(x) \leq f(d)$ for all x near d.

$f(a)$ is smallest value of $f(x)$ on all of $[a,b]$ – this is the absolute minimum of $f(x)$ on $[a,b]$

i.e. $f(x) \geq f(a)$ for all x in $[a,b]$

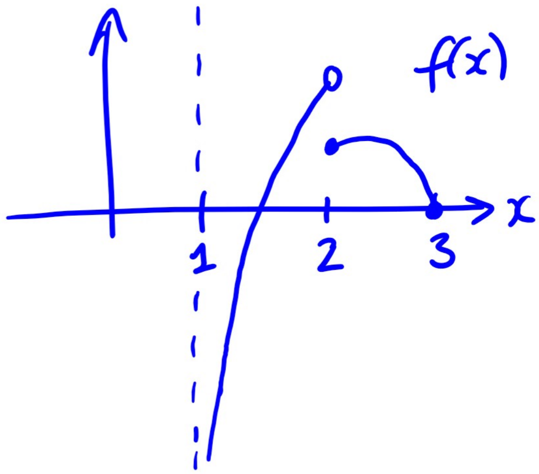
"Local" max./min. only happen inside interval not at endpoints.



← Absolute max. = 1

← Abs. min. = −1

In general $f(x)$ could "attain" abs. max. (or min.) at more than one x-value.



NO absolute maximum — you want it to be $\lim_{x\to 2^-} f(x)$ but this is not a value taken by f at any x-value, in particular not equal to $f(2)$.

NO absolute minimum ($-\infty$ not a value $f(x)$ takes!)

Extreme Value Theorem

Extreme value = abs. max./min.

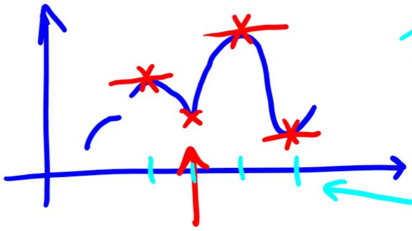
If $f(x)$ is continuous on a closed interval $[a,b]$, then $f(x)$ attains an absolute maximum and an absolute minimum.

(Does not tell us how [abs] many [places we achieve each of the absolute maximum or minimum].)

But how do we find them? (↳ The absolute min. & max.)

Notice If $f(x)$ is continuous on $[a,b]$, then abs. max. (or min.) can only be found at points with local max. (or min.) or at endpoints.

Fermat's theorem If $f(x)$ has a local max. (or min.) at $x=c$ then $f'(x)$ is not defined at $x=c$ or $f'(c)=0$.



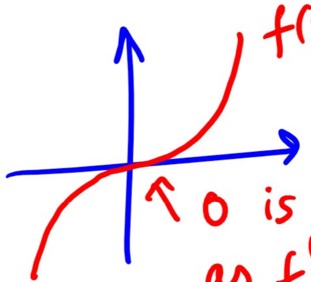
The red x points correspond to critical numbers (which are x-values).

f' not defined (think $y=|x|$ at $x=0$)

(All of the red x points are local maxima or minima here — can also have discontinuities as critical number.)

Definition A critical number of $f(x)$ is any x-value c in $\mathrm{dom}(f)$ for which we have $f'(x)$ not defined at $x=c$ or $f'(c)=0$.

So Fermat's Theorem tells us that to search for local max./min., enough to check critical numbers.



↑ 0 is a critical # as $f'(0)=0$ BUT $0=f(0)$ not a local max./min.

← WARNING!! So all local max/min. happen at critical numbers, but not every critical number gives a local max./min!

So our strategy for searching for absolute maximum/minimum of $f(x)$ which is continuous on $[a,b]$ is as follows:

Closed Interval Method

1. Find $f(c)$ for every critical number in $[a,b]$.
2. Find $f(a)$ and $f(b)$.
3. The biggest value among all values of $f(x)$ in 1 and 2 is the absolute max. and the smallest is the absolute minimum.

__Example__ Find abs. max./min. of $f(x) = x^{2/3}(6-x)$ on $[-1, 6]$.

__Solution__ (For now we trust $f(x)$ continuous.)

① Find critical #s of $f(x)$:

$$\text{Find } f'(x) = \left(6x^{2/3} - x^{5/3}\right)'$$
$$= 6\left(\tfrac{2}{3}\right)x^{-1/3} - \left(\tfrac{5}{3}\right)x^{2/3}$$
$$= 4x^{-1/3} - 5/3\, x^{2/3}$$
$$= \frac{4 - 5/3\, x}{x^{1/3}}$$

So $f'(x) = 0$ when $4 - 5/3\, x = 0$ i.e. $\underline{\underline{x = \frac{12}{5}}}$

and $f'(x)$ is not defined when $x^{1/3} = 0$ i.e. $\underline{\underline{x = 0}}$

We need $f\left(\frac{12}{5}\right) = \left(\frac{12}{5}\right)^{2/3}\left(6 - \frac{12}{5}\right) \approx \boxed{6.45}$

& $f(0) = 0^{2/3}(6-0) = \boxed{0}$

② Find $f(-1) = (-1)^{2/3}(6-(-1)) = \boxed{7}$

& $f(6) = 6^{2/3}\cancel{(6-6)}^{0} = \boxed{0}$.

③ Compare: abs. max. $= 7$

abs. min. $= 0$