1A03 – Calculus I for Science
(Section C02)

Lecture 15

Last time: Maximum & Minimum Values

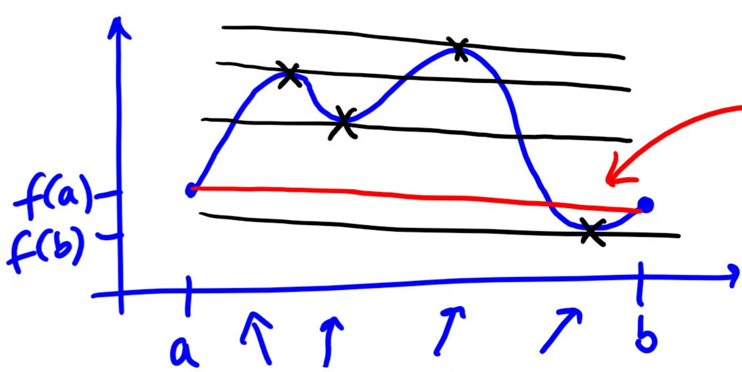
$f(x)$ on $[a,b]$ has:

an **absolute** maximum at $x=c$ if $f(x) \le f(c)$ — for all x
an **absolute** minimum at $x=c$ if $f(x) \ge f(c)$ — for all x

a **local** maximum at $x=c$ if $f(x) \le f(c)$ — for all x near c
a **local** minimum at $x=c$ if $f(x) \ge f(c)$ — for all x near c

absolute max. & min. can only happen at endpoints a, b or CRITICAL NUMBERS: c with $f'(c) = 0$ or undefined.

4.2 Mean Value Theorem



slope $\dfrac{f(b) - f(a)}{b-a}$ — "average rate of change of $f(x)$ over $[a,b]$"

Mean Value Theorem (MVT)

Let $f(x)$ be a function with ① $f(x)$ continuous on $[a,b]$
② $f(x)$ differentiable on (a,b)

There is some $c \in [a,b]$ with
$$f'(c) = (f(b) - f(a)) / (b-a).$$

Special Case: Rolle's theorem: If $f(a) = f(b)$ in MVT then there is some $c \in [a,b]$ with $f'(c) = 0$.

Example Find c satisfying the conclusion of the MVT for $f(x) = \dfrac{x}{x+3}$ on $[1,2]$.

Solution (We should check $f(x)$ continuous on $[1,2]$ & differentiable on $(1,2)$ — but we'll assume that here.)

MVT: there is $c \in [1,2]$ with
$$f'(c) = \frac{f(2)-f(1)}{2-1} = 2/5 - 1/4 = \frac{3}{20}.$$

Now find
$$f'(x) = \left(x(x+3)^{-1}\right)'$$
$$= (x+3)^{-1} - x(x+3)^{-2} = \frac{1}{x+3} - \frac{x}{(x+3)^2}$$
$$= \frac{x+3-x}{(x+3)^2} = \frac{3}{(x+3)^2}$$

So now solve for c in $f'(c) = \dfrac{3}{20}$

i.e. $\dfrac{3}{(c+3)^2} = \dfrac{3}{20}$ i.e. $(c+3)^2 = 20$

$\Rightarrow c+3 = \sqrt{20}$

$\Rightarrow c = \sqrt{20} - 3$

$\approx 1.47.$

4.3 Derivatives & Graph Shape

Increasing / Decreasing Test

Continue to assume for this Test that $f(x)$ is as in MVT.

(It does not make much sense to say that $f'(x) > 0$ or $f'(x) < 0$ if f' not defined!)

(a) If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on that interval

(b) If $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on that interval.

[Call our interval $[a,b]$. Somewhere in $[a,b]$, c say, we have

$$\underset{\text{in (a)}}{0 <} \overset{\text{in (b)}}{>} f'(c) = \frac{f(b) - f(a)}{b - a}$$

i.e. $f(b) > f(a)$

$f(b) < f(a)$ in (b).]

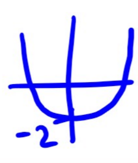
(b) similarly reasoned

So the only places where $f(x)$ can switch from increasing to decreasing (or vice versa) are when $f'(x) = 0$.
(We're assuming $f(x)$ differentiable to apply MVT.)

Example Find where $f(x) = x^6 - 3x^4 + 3$ is increasing and decreasing.

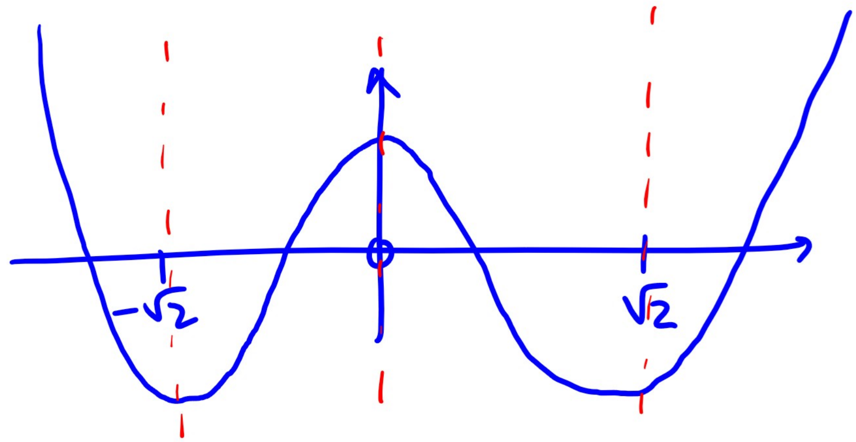
Solution Separate the real line into intervals on which $f(x)$ is increasing / decreasing using x-values for which $f'(x) = 0$.

Solve for x in
$$0 = f'(x) = 6x^5 - 12x^3$$
$$= 6x^3(x^2 - 2)$$
$$\Rightarrow x = 0, \pm\sqrt{2}.$$



Four intervals:	$f'(x)$	$6x^3$	x^2-2	inc/dec
$(-\infty, -\sqrt{2})$	$-$	$-$	$+$	dec.
$(-\sqrt{2}, 0)$	$+$	$-$	$-$	inc.
$(0, \sqrt{2})$	$-$	$+$	$-$	dec.
$(\sqrt{2}, \infty)$	$+$	$+$	$+$	inc.

[For curiosity:



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For $f(x)$ continuous:

Notice that we are not assuming $f(x)$ differentiable; this test also works for c where $f'(c)$ undefined.

First derivative Test

If c is a critical # and $f'(x)$ changes from negative to positive at $x=c$, we have a local minimum & if $f'(x)$ changes from positive to negative at $x=c$, we have a local maximum.

Notice If $f'(x)$ is negative both sides of $x=c$ (or positive " " " ") we do NOT have a local max. or min.

Second Derivative Test — coming next; packages these ideas together.