1A03 — Calculus I for Science
(Section C02)

Lecture 16

Last time

Increasing/Decreasing Test

If $f(x)$ is differentiable and $f'(x) \underset{<}{>} 0$ on an interval, then $f(x)$ is increasing / decreasing on that interval

First Derivative Test

If $f(x)$ is continuous, c a critical # of $f(x)$, then we have a local maximum / minimum at c if $f'(x)$ changes at c from $+ve$ to $-ve$ / $-ve$ to $+ve$.

Put these together:

Second Derivative Test

If $f''(x)$ is continuous near c with $f'(c) = 0$, then

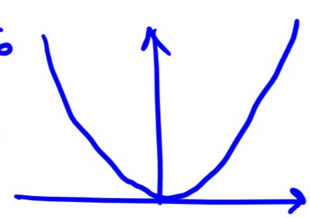
(a) if $f''(c) > 0$, ← f' increasing, then f has a local minimum at $x = c$

(b) if $f''(c) < 0$ ← f' decreasing, then f has a local maximum at $x = c$.

Notice This test does not tell us what we have at $x = c$ if

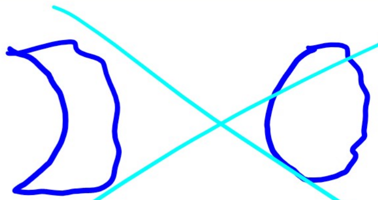
$$f''(c) = 0.$$

e.g. $f(x) = x^6$ at $x = 0$



The 2nd derivative also tells us something about $f(x)$ even away from critical #s:

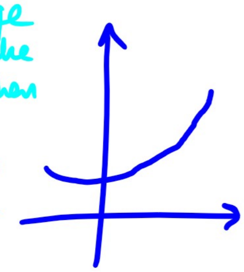
Concavity



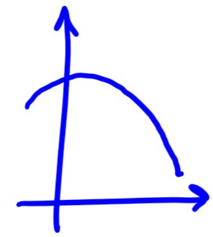
concave convex

Normal usage does not make sense here when no "inside" or "outside" to the graph.

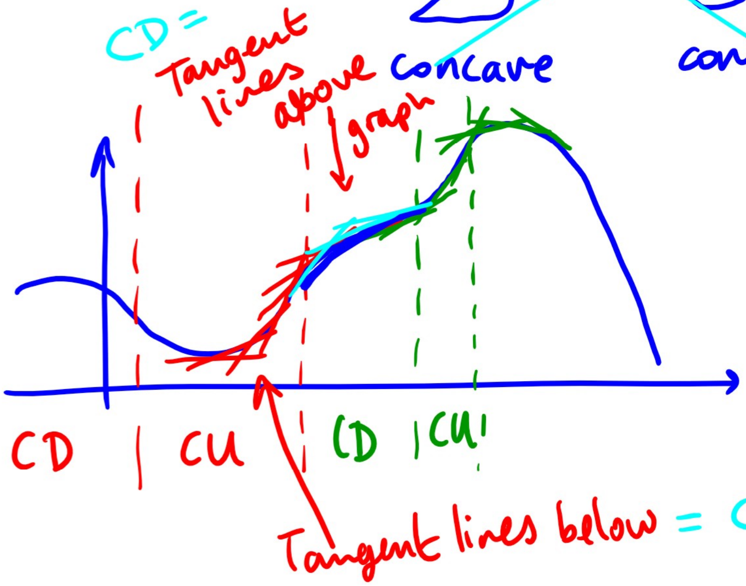
So instead:



Concave upward = CU



concave downward = CD



CONCAVITY TEST

If $f''(x) > 0$ on an interval, then the graph of $f(x)$ is concave upward on that interval

If $f''(x) < 0$ on an interval, then the graph of $f(x)$ is concave downward on that interval

A point of inflection $(x, f(x))$ is a point at which the concavity changes (from CU to CD, or CD to CU).

(So in particular $f''(x) = 0$ (or is undefined, but not infinite) but not necessarily $f'(x) = 0$.)

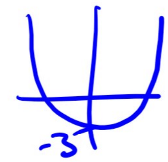
Example Where is the graph of $f(x) = x^5 - 10x^3 + 7$ CU, CD & where are the points of inflection?

Solution $f(x)$ is a polynomial, so all derivatives defined, thus: First find x with $f''(x) = 0$. (Don't need to worry about f'' undefined.)

$$f'(x) = 5x^4 - 30x^2$$

So $f''(x) = 20x^3 - 60x = 20x(x^2-3)$.

So $f''(x)=0$ when $x = 0, \pm\sqrt{3}$.



Intervals	$f''(x) \gtrless 0$?	? CU / CD	$20x$	x^2-3
$(-\infty, -\sqrt{3})$	$-$	CD	$-$	$+$
$(-\sqrt{3}, 0)$	$+$	CU	$-$	$-$
$(0, \sqrt{3})$	$-$	CD	$+$	$-$
$(\sqrt{3}, \infty)$	$+$	CU	$+$	$+$

$\leftarrow -\sqrt{3}$

$\leftarrow 0$

$\leftarrow \sqrt{3}$

all points of inflection.

as concavity changes each time

$$f(x)\Big|_{x=0} = \frac{\sin x}{e^x - 1}\Big|_{x=0} = \frac{0}{0} \quad ??$$

What about $\displaystyle\lim_{x\to 0}\left(\frac{\sin x}{e^x - 1}\right)$??

If we have $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)}$ with $\displaystyle\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$,

then this limit is an <u>indeterminate form of type $\frac{0}{0}$</u> — get stuck with limit laws.

Same problem with an <u>indeterminate form of</u>

type $\frac{\infty}{\infty}$, where $\lim_{x \to a} f(x) = \pm\infty$

and $\lim_{x \to a} g(x) = \pm\infty$.

Don't have to be equal e.g. $\frac{-\infty}{\infty}$ OK.

How do we determine which one wins, or if they compromise on a #?

Let's look again at $\lim_{x \to 0} \frac{\sin(x)}{3(e^x - 1)}$ – zoom in on $x = 0$

& approximate $\sin x$ and $3(e^x - 1)$ by their tangent lines at $x = 0$:

We could also have done this with $\lim_{x \to 0} \frac{\sin x}{e^x - 1}$, and approximated $\frac{\sin x}{e^x - 1}$ by $\frac{x}{x} = 1$ near 0 & then expect $\lim_{x \to 0} \frac{\sin x}{e^x - 1} = 1$.

$$\sin x \approx (\sin(x))'(0) \cdot x = \cos(0) \cdot x = x$$

$$3(e^x - 1) \approx (3(e^x - 1))'(0) \cdot x = 3e^0 \cdot x = 3x$$

So $\frac{\sin x}{3(e^x - 1)} \approx \frac{x}{3x} = \frac{1}{3}$ near $x = 0$

So we expect (now) that $\lim_{x \to 0} \frac{\sin x}{3(e^x - 1)} = \frac{1}{3}$.

The example in class was meant to illustrate more transparently what is happening because not everything cancels in the end.

This kind of reasoning gives the following rule:

(4.4) L'Hospital's Rule

If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$

or $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$, then

i.e. this could be indet. form of type $\frac{0}{0}$ or ∞/∞.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

as long as this new limit exists.

(a could be any # or $\pm\infty$, and this also works for one-sided limits i.e. can replace a by a^+ or a^-)

<u>Example</u> Find $\lim_{x \to 0} \left(\frac{e^{2x} - 1}{x} \right)$.

<u>Solution</u> $\lim_{x \to 0} (e^{2x} - 1) = \lim_{x \to 0} x = 0$ so

L'Hospital applies & we get

$$\lim_{x \to 0} \left(\frac{e^{2x} - 1}{x} \right) = \lim_{x \to 0} \left(\frac{2e^{2x}}{1} \right) = 2.$$