1A03 – CALCULUS I FOR SCIENCE
(SECTION C02)

Lecture 17

Last time   L'HOSPITAL'S RULE

If we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$,

i.e. $\lim_{x\to a} \frac{f(x)}{g(x)}$ where $\boxed{0 = \lim_{x\to a} f(x) = \lim_{x\to a} g(x)}$ OR $\boxed{\lim_{x\to a} f(x) = \pm\infty \text{ AND } \lim_{x\to a} g(x) = \pm\infty}$

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ ← as long as this limit exists or is $\pm\infty$.

Example Find $\lim_{x\to\infty} \left(\frac{x - e^x}{x^2}\right)$.

Solution $\lim_{x\to\infty} (x - e^x) = -\infty$ & $\lim_{x\to\infty} x^2 = \infty$

Why? Properly, this is justified formally by the end of this lecture (below) – indeterminate form of type $\infty - \infty$

So we have "form $\frac{\infty}{\infty}$" so L'Hospital applies:

$$\lim_{x\to\infty} \left(\frac{x-e^x}{x^2}\right) = \lim_{x\to\infty}\left(\frac{1-e^x}{2x}\right) = \frac{-\infty}{\infty}$$

So we can apply L'Hospital again!

We get $\lim_{x\to\infty}\left(\frac{-e^x}{2}\right) = -\infty$.

Example Find $\lim_{x\to\frac{\pi}{2}} \left(\frac{1-\sin(x)}{\csc(x)}\right)$.

Solution $\lim_{x\to\pi/2} (1 - \sin(x)) = 0$ & $\lim_{x\to\pi/2} \csc(x) = \lim_{x\to\frac{\pi}{2}} \frac{1}{\sin(x)} = 1$

WARNING

L'HOSPITAL DOES NOT APPLY HERE!!!

But we can compute the limit here: $\lim_{x\to\pi/2} \left(\frac{1-\sin(x)}{\csc(x)}\right) = \frac{0}{1} = 0$.

In a fight between 0 and a #, 0 will always win: $\frac{0}{\#} = 0$, $\frac{\#}{0} = \pm\infty$

Example Find $\lim_{x\to -\infty} x e^{x}$

Solution $\lim_{x\to -\infty} xe^{x}$ is of the form $-\infty . 0$

Idea : If we have $\lim_{x\to a} f(x)g(x)$ where

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty$$

(this limit is then called an "indeterminate product")

We can rewrite this as

$$\lim_{x\to a} \left(\frac{f(x)}{(1/g(x))}\right) \quad \text{or} \quad \lim_{x\to a} \frac{g(x)}{(1/f(x))}$$

$\to$ form $\frac{0}{0}$ $\qquad\qquad$ $\to$ form $\frac{\infty}{\infty}$

So apply L'Hospital.

Back to Example: $\lim_{x\to -\infty} x . e^{x} = \lim_{x\to -\infty} \frac{e^{x}}{(1/x)}$

Practice will help you to know which option is more likely to work — try the other one though if things don't work out with your first choice.

$$\overset{\text{L'H. } \frac{0}{0}}{=} \lim_{x \to -\infty} \frac{e^x}{-x^{-2}} \leftarrow \text{form } \frac{0}{0}$$

→ I'm making things worse somehow — try other version:

$$\lim_{x \to -\infty} x e^x = \lim_{x \to -\infty} \frac{x}{1/e^x}$$

$$\overset{\text{L'H. } \frac{\infty}{\infty}}{=} \lim_{x \to -\infty} \frac{1}{-e^{-x}} = \frac{1}{\infty} = 0.$$

Example Find $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$.

Solution This has indeterminate form 1^∞.

$\lim_{x\to\infty}\left(1+\frac{1}{x}\right) = 1$; $\infty \leftarrow \lim_{x\to\infty} x$

Idea : If $\lim_{x \to a} f(x)^{g(x)}$ has form 1^∞, or 0^0, or ∞^0

then rewrite this limit as

$$\lim_{x \to a} e^{\ln\left(f(x)^{g(x)}\right)} = \lim_{x \to a} e^{g(x) \cdot \ln f(x)}$$

$$= e^{\underline{\lim_{x \to a} g(x) \cdot \ln f(x)}}$$

↑ Use L'Hospital to deal with this.

In each indeterminate power case ... we get an indeterminate product in the exponent

1^∞ : $\infty \cdot \ln 1 = \infty \cdot 0$

0^0 : $0 \cdot \ln 0 = 0 \cdot -\infty$

∞^0 : $0 \cdot \ln \infty = 0 \cdot \infty$

Back to example :

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^{\underbrace{\lim_{x\to\infty} x \cdot \ln\left(1+\frac{1}{x}\right)}}$$

This exponent limit $= \displaystyle\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\left(1/x\right)}$

$$\underset{\text{L'H. } \frac{0}{0}}{=} \lim_{x\to\infty} \frac{\left(-1/x^2\right)/\left(1+1/x\right)}{-1/x^2}$$

$$= \lim_{x\to\infty} \frac{1}{1+1/x} = 1$$

So $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^1 = e.$

In other words, the two warring factions in 1^∞ compromise & we get a # instead of 1 or ∞.
$\neq 1$

Example Find $\displaystyle\lim_{x\to 0}\left(\cot(x) - \frac{1}{x}\right)$.

Solution This has indeterminate form of type $\infty - \infty$.

Idea Try to convert a $\infty - \infty$ form to a quotient or product & if that then has indeterminate form, then apply L'Hospital.

Back to Example: $\displaystyle\lim_{x\to 0}\left(\cot(x) - \frac{1}{x}\right)$

$$\circledast = \lim_{x\to 0}\left(\frac{\cos(x)}{\sin(x)} - \frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{x\cos(x) - \sin(x)}{x\sin(x)}\right)$$

See this symbol below.

$$\underset{\text{L'H. } \frac{0}{0}}{=} \lim_{x\to 0}\left(\frac{-x\sin(x) + \cos(x) - \cos(x)}{x\cos(x) + \sin(x)}\right)$$

$$= \lim_{x\to 0}\left(\frac{-x\sin(x)}{x\cos(x) + \sin(x)}\right)$$

$$\underset{\text{L'H. } \frac{0}{0}}{=} \lim_{x\to 0}\left(\frac{-x\cos(x) - \sin(x)}{-x\sin(x) + \underbrace{\cancel{\cos(x) + \cos(x)}}_{2\cos(x)}}\right) = \frac{-0-0}{-0+2} = \frac{0}{2}$$

$$= 0$$

Exercise Find $\lim_{x\to 0}\left(\tan(2x)\right)^{x}$.

Answer to be posted next week.

$\circledast$ I was asked after class, why not say

$$\lim_{x\to 0}\left(\cot(x) - \frac{1}{x}\right) = \lim_{x\to 0}\frac{x\cot(x) - 1}{x}$$ and proceed?

Well, let's see how we can proceed! Of course $\lim_{x\to 0} x = 0$.

Now, $\lim_{x\to 0}\left(x\cot(x) - 1\right) = \underbrace{\left(\lim_{x\to 0} x\cot(x)\right)}_{} - 1$

What is this limit? Well, $\lim_{x\to 0} x = 0$, $\lim_{x\to 0}\cot(x) = \infty$

So we have an indeterminate product to check:

$$\lim_{x\to 0} x\cot(x) = \lim_{x\to 0}\left(\frac{x}{1/\cot(x)}\right) = \lim_{x\to 0}\frac{x}{\tan x}$$

$$\underset{\text{L'H. } \frac{0}{0}}{=} \lim_{x\to 0}\frac{1}{\sec^2(x)} = \lim_{x\to 0}\cos^2(x) = 1.$$

That means $\lim_{x\to 0}\left(x\cot(x) - 1\right) = 0.$

So now we know that our original limit $\lim_{x\to 0}\left(\frac{x\cot(x)-1}{x}\right)$ is in indeterminate form of type $\frac{0}{0}$, so we can apply L'Hospital:

$$\lim_{x\to 0}\left(\frac{x\cot(x)-1}{x}\right) \overset{\text{L'H. } \frac{0}{0}}{=} \lim_{x\to 0}\left(\frac{\cot(x) - x\csc^2(x)}{1}\right)$$

$$= \lim_{x\to 0}\left(\cot(x) - x\csc^2(x)\right).$$

Now, what is this? $\lim_{x\to 0}\cot(x) = \infty$, as above.

$$\lim_{x\to 0} x\csc^2(x) = \lim_{x\to 0}\frac{x}{\sin^2(x)} \underset{\text{L'H. } \frac{0}{0}}{=} \lim_{x\to 0}\frac{1}{2\sin(x)\cos(x)} = \frac{1}{0} = \infty.$$

So our $\lim_{x\to 0}\left(\cot(x) - x\csc^2(x)\right)$ is again an indeterminate difference $\infty - \infty$.

We need to evaluate it by making it into a suitable quotient or product.

$$\lim_{x\to 0}\left(\cot(x) - x\csc^2(x)\right) = \lim_{x\to 0}\left(\frac{1}{\tan(x)} - \frac{x}{\sin^2(x)}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin^2(x) - x\tan(x)}{\tan(x)\sin^2(x)}\right)$$, say. (I had to make a choice.)

$$\overset{\text{L'H } \frac{0}{0}}{=} \lim_{x\to 0}\left(\frac{2\sin(x)\cos(x) - \tan(x) - x\sec^2(x)}{\underset{\tan^2(x)}{\cancel{\sec^2(x)\sin^2(x)}} + \underset{2\sin^2(x)}{\cancel{2\tan(x)\sin(x)\cos(x)}}}\right)$$

$$\underset{\text{L'H } \frac{0}{0}}{=} \lim_{x\to 0}\left(\frac{\overbrace{-2\sin^2(x)}^{0} + \overbrace{2\cos^2(x)}^{2}\ \overbrace{-2\sec^2(x)}^{-2} - \overbrace{2x\sec^2(x)\tan(x)}^{0}}{2\tan(x)\sec^2(x) + 4\sin(x)\cos(x)}\right)$$

$$\underset{\text{L'H } \frac{0}{0}}{=} \lim_{x\to 0}\left(\frac{\overbrace{-8\sin(x)\cos(x)}^{0} - \overbrace{6\sec^2(x)\tan(x)}^{0} - \overbrace{4x\sec^2(x)\tan^2(x)}^{0} - \overbrace{2x\sec^4(x)}^{0}}{\underbrace{2\sec^4(x)}_{2} + \underbrace{4\tan^2(x)\sec^2(x)}_{0} + \underbrace{4\cos^2(x)}_{4} - \underbrace{4\sin^2(x)}_{0}}\right)$$

$$= \frac{0}{6} = \underline{\underline{0}}.$$

So we do have the tools to proceed in this way! But it is not only a more complicated limit to evaluate at the end (with more derivative iterations), we also twice needed to use L'Hospital just to check that we could apply L'Hospital! So some routes worse... but not invalid.