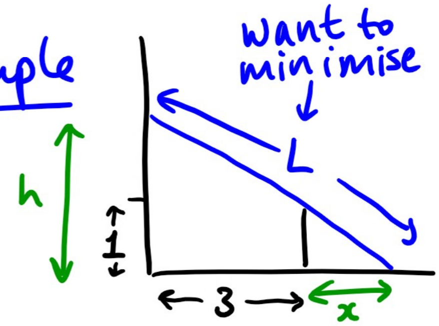
1A03 – CALCULUS I FOR SCIENCE
(SECTION C02) Lecture 20

Last time OPTIMIZATION PROBLEMS

→ Turning word problems into equations
→ Solution comes from maximising or minimising some $f(x)$
↳ finding absolute max./min.

Example



From $h^2+(x+3)^2=L^2$ and $\frac{x+3}{h}=\frac{x}{1}$ we saw

$$L^2 = (x+3)^2\left(\frac{1}{x^2}+1\right)$$ ← a function of 1 variable

We want to minimise L, so we could write

$$L=(x+3)\left(\frac{1}{x^2}+1\right)^{1/2}$$ & minimis~~ing~~e

or (perhaps this is easier) minimise L^2 (& $\sqrt{\ }$ at end)

Domains of variables: $L>0$, $x>0$, say $h>0$
(or $h>1$ — no longer relevant however)

$x \in (0,\infty)$

Check critical values:

$$(L^2)' = 2(x+3)\left(\frac{1}{x^2}+1\right) + (x+3)^2\left(-\frac{2}{x^3}\right)$$

$$= \dots = \frac{2(x+3)(x^3-3)}{x^3}$$

Where is $(L^2)'$ undefined? Nowhere ($x>0$): defined everywhere.

Where is $(L^2)'=0$: when $x^3-3=0$ (and $x>0$)

i.e. $x = \sqrt[3]{3}$.

First deriv. test : $(L^2)' < 0$ on $(0, \sqrt[3]{3})$ L^2 dec.

$(L^2)' > 0$ on $(\sqrt[3]{3}, \infty)$ L^2 inc.

min.

So at $x = \sqrt[3]{3}$ we have a local min.

So min. of L^2 occurs at $x = \sqrt[3]{3}$ (no other poss.)

If there's just one critical value and it's a local min., everywhere else L^2 must be bigger.

i.e. $L^2 \cong 33.42$

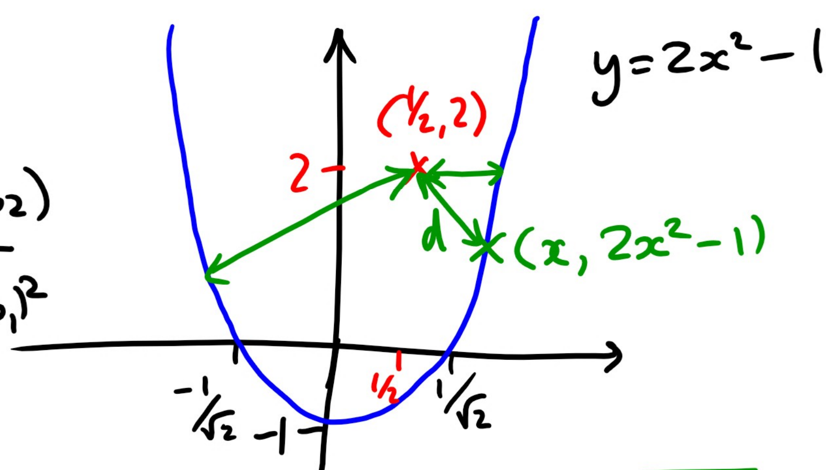
So min. val. of $L \approx 5.78\,m$.

Example Find the point on the parabola $y = 2x^2 - 1$ closest to the point $(\frac{1}{2}, 2)$.

Solution

Distance between (a_1, b_1) and (a_2, b_2)

$= \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2}$



$$d = \sqrt{(x - \tfrac{1}{2})^2 + (2x^2 \underbrace{-1-2}_{-3})^2}$$

Minimise d

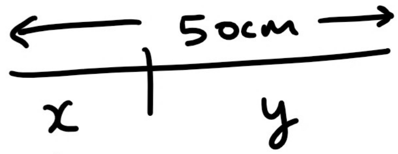
... or, if you prefer, could minimise d^2

(then $\sqrt{\ }$ at end)

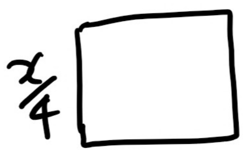
Exercise (If you're feeling a little brave ...)

Example A wire of 50cm is cut into 2 pieces, with one piece used to make a square, the other piece used to make a circle. Where should the wire be cut to maximise the TOTAL area of the 2 shapes?

Solution



$x + y = 50$



Circumference $= y$

$= 2\pi r$

Perimeter $= x$

Area $= \left(\frac{x}{4}\right)^2$

Area $= \pi r^2 = \pi\left(\frac{y}{2\pi}\right)^2$

$$\text{Total area } A = \left(\frac{x}{4}\right)^2 + \pi\left(\frac{y}{2\pi}\right)^2$$

$$= \frac{x^2}{16} + \frac{y^2}{4\pi} = \frac{x^2}{16} + \frac{(50-x)^2}{4\pi}$$

eliminate $y = 50 - x$

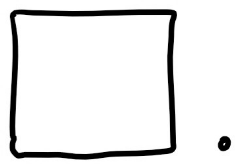
Differentiate: $A' = \frac{x}{8} - \frac{(50-x)}{2\pi}$

Where is $A' = 0$? When $= 0$; $x = \frac{200}{4+\pi}$ ... check

≈ 28 cm

Then $A \cong 87.5 \text{ cm}^2$

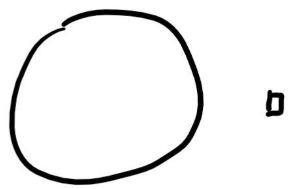
What happens if we make a really big square & a tiny circle



so $x \approx 50$
$y \approx 0$

? Then what is $A \approx \left(\frac{50}{4}\right)^2 = 156.25\,cm^2$

What happens if we make a really big circle and a tiny square

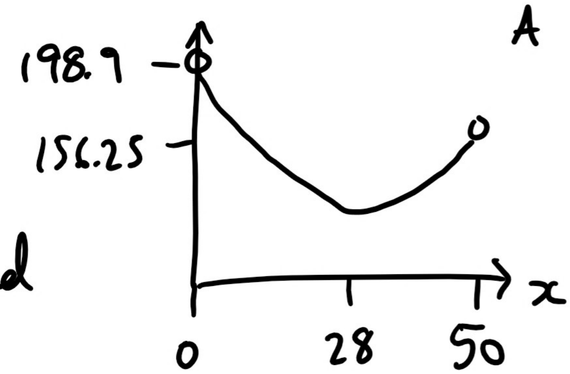


so $y \approx 50$, $x \approx 0$

Then $A \approx \pi\left(\frac{50}{2\pi}\right)^2 = 198.9\,cm^2$.

What's going on here: we found minimum

Bad question: there is no max. We would only achieve max. if we were allowed to use all wire for circle



Be careful: just because there is one critical point does not mean what you want to happen happens there!!

Remark: In the text book there are similar problems (4.7 Ex. 37,38) which, judging from answers, allow all wire to be used for one shape; I think this is badly worded, and if there is supposed to be a cut then we cannot use all the wire for one shape. Rest assured we will do our best to make sure test + assignment problems are not ambiguous!

4.9 Antiderivatives

Undoing — as much as we can — the act of differentiating a function.

Definition Given a function $f(x)$ on interval I an antiderivative $F(x)$ is a function with

$$\frac{dF}{dx} \to F'(x) = f(x) \qquad \text{for all } x \in I.$$

integral

We have notation $F(x) = \int f(x)\,dx$.

We also call an antiderivative : an indefinite integral.

(Later we'll see what a "definite integral".)