1A03 – CALCULUS I FOR SCIENCE
(SECTION C02)

Lecture 21

Last time: ANTIDERIVATIVES

Given $f(x)$ (defined on $x \in I$), an antiderivative of $f(x)$ is any function $F(x)$ with $F'(x) = f(x)$. (for all x in I)

Another name for antiderivative: indefinite integral

Notation: $F(x) = \int f(x)\,dx$.

Exercise Write down an antiderivative of $f(x) = x^5$.

Possible Solutions: $\frac{1}{6}x^6$ $\qquad$ $\frac{1}{6}x^6 + 5$ $\qquad$ $\frac{1}{6}x^6 + 17$

Can we have an antiderivative of $f(x) = x^5$ that doesn't look like $(1/6)x^6 + \text{constant}$?

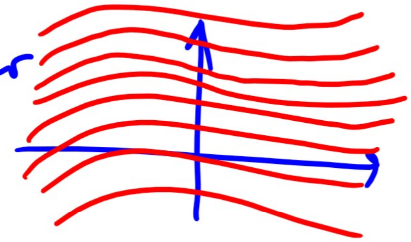
Suppose we have a function $f(x)$ with antiderivatives $F(x)$ and $G(x)$.

$$\frac{d}{dx}\left(F(x) - G(x)\right) = F'(x) - G'(x)$$
$$= f(x) - f(x) = 0$$

So (think back to inc./dec. test & MVT) we have $F(x) - G(x) = \text{constant}$ i.e.

$$F(x) = G(x) + \text{constant}.$$

So all antiderivatives of a given function $f(x)$ are vertical translates of one another



If we find one antiderivative $F(x)$ of $f(x)$, then we have the whole "family" of antiderivatives $F(x) + C$.

(Beginning) Rules for finding antiderivatives of complicated functions using constituent parts:

→ we will get to product rule, chainrule etc.

Reversing Sum/Difference & constant multiples rules:

Suppose
$$F(x) = \int f(x)\,dx$$
$$G(x) = \int g(x)\,dx$$

Then
$$F(x) \pm G(x) = \int (f(x) \pm g(x))\,dx$$
$$= \int f(x)\,dx \pm \int g(x)\,dx$$

And if k is a real constant, then
$$kF(x) = \int k f(x)\,dx$$
$$= k\int f(x)\,dx$$

(See tables on p. 352 & p. 403 of textbook.)

Function $f(x)$	Antiderivatives $F(x) = \int f(x)\,dx$
x^n	$\dfrac{x^{n+1}}{n+1} + C$
1	$x + C$
$\cos(x)$	$\sin(x) + C$
$\sin(x)$	$-\cos(x) + C$
$\sec^2(x)$	$\tan(x) + C$
$\sec(x)\tan(x)$	$\sec(x) + C$
e^x	$e^x + C$
b^x	$\dfrac{1}{\ln b} b^x + C$
$\dfrac{1}{1+x^2}$	$\arctan(x) + C$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin(x) + C$
$\sinh(x)$	$\cosh(x) + C$
$\cosh(x)$	$\sinh(x) + C$
$\dfrac{1}{x}$	$\ln\lvert x\rvert + C$

$$\int \frac{1}{x}\,dx = \ln(x) + C \quad \text{if } x > 0$$

What if $x < 0$? Look at $\dfrac{d}{dx}\left(\ln(-x)\right) = \dfrac{-1}{-x} = \dfrac{1}{x}$

So $\displaystyle\int \frac{1}{x}\,dx = \ln(-x) + C$ for $x < 0$.

What is $|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$?

Example Find all antiderivatives of

$$f(x) = \frac{6x^7 + \sqrt[3]{x}}{x^2} - 2\sec^2 x$$

Solution Simplify: $f(x) = 6x^5 + x^{1/3 - 2} - 2\sec^2 x$

$$= 6x^5 + x^{-5/3} - 2\sec^2 x$$

(Using table) we have

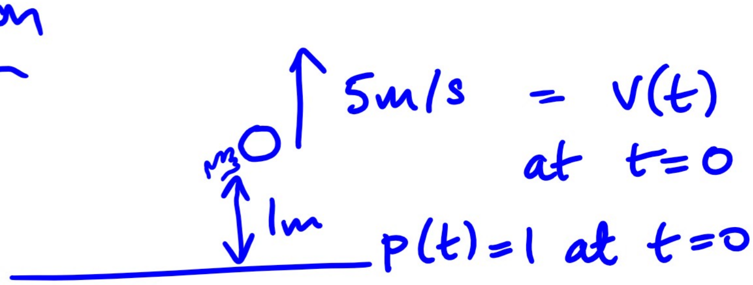
$$\int f(x)\,dx = x^6 + C + \frac{x^{-5/3+1}}{(-5/3+1)} + C - 2\tan(x) + C$$

$$+ C$$

$$= x^6 - \frac{3}{2}x^{-2/3} - 2\tan(x) + C.$$

Example (Initial Value Problem) Throw mathematical (spherical & frictionless) aardvark up in the air with an initial speed of 5 m/s from a height of 1m. Find position and velocity as a function of time t.

Solution



We also have

$$a(t) = -g$$
$$= -10$$

Since $v'(t) = a(t)$

we have $v(t) = \int a(t)\,dt = \int -10\,dt = -10 \int 1\,dt$

$$= -10t + C$$

We also know $v(0) = 5$ so can find C

Since we have an initial value for the velocity, we get to identify which one of the possible antiderivative family is the relevant one in this case.

$$5 = -\underbrace{10 \cdot 0}_{0} + C \Rightarrow C = 5$$

For us $\underline{v(t) = -10t + 5}$.

Since $p'(t) = v(t)$,

$$p(t) = \int v(t)\,dt = \int -10t + 5\,dt$$

$$= -10\int t\,dt + 5\int 1\,dt$$

$$= \frac{-10t^2}{2} + 5t + C$$

Again use initial values to solve for C:

Similarly, we can use the initial data (value of $p(0)$) to pick out the one antiderivative from the family that is relevant here.

$$p(0) = 1 = -5\underbrace{\cdot 0^2}_{0} + 5\underbrace{\cdot 0}_{0} + C$$

$$\Rightarrow C = 1.$$

So $\underline{p(t) = -5t^2 + 5t + 1}$.

Part B When does the aardvark hit the ground?

Solution i.e. for which t is $p(t) = 0$?

Solve $-5t^2 + 5t + 1 = 0$

Quadratic formula: $t = \dfrac{-5 \pm \sqrt{25 + \cancel{100}\ 20}}{-\cancel{2}\ 10}$

$\Rightarrow t \simeq \cancel{\dfrac{-5}{2} \pm \dfrac{11.38}{2}}$ $\quad -\dfrac{1}{2} \pm \dfrac{\sqrt{45}}{10}$

Sorry about the computational error — fortunately you all know how to solve the quadratic formula!!

Time here runs forwards from $t=0$

So only +ve solution is valid

i.e. $t \simeq \cancel{3.09}\,\mathrm{s}$, $1.17\,\mathrm{s}$