1A03 – Calculus I for Science

(Section C02)

Lecture 22

Today: Sigma Notation : Writing Sums

→ compact way of writing sums

Example

$$1 + 4 + 9 + 16 + 25 + 36$$

$$= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2$$

$$= \sum_{i=1}^{6} i^2$$

"The sum of i squared from i equals 1 to 6"

index

plug in & successive terms of sum have i increase by 1 each time

Example

$$\sum_{i=1}^{3} \frac{1}{i} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3}$$

Example

$$\sum_{k=7}^{11} 2^k = 2^7 + 2^8 + 2^9 + 2^{10} + 2^{11}$$

index not always i

Example

$$\sum_{k=0}^{5} 3 = \underset{k=0}{3} + \underset{k=1}{3} + \underset{k=2}{3} + \underset{k=3}{3} + \underset{k=4}{3} + \underset{k=5}{3}$$

Example

$$\sum_{i=1}^{n} (i-1) = (1-1) + (2-1) + (3-1) + \ldots + ((n-1)-1) + (n-1)$$

__Warning:__ don't get mixed up between which letter is index (changing each term) & any other unknown letters

Go back to last Example: $\ldots = 0 + 1 + 2 + \ldots + (n-2) + (n-1)$

$$= \sum_{i=0}^{n-1} i$$

i.e. that's 2 different ways to write the same sum!!!

__Index-Shifting:__

__Example__ $\displaystyle\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 + n^2$

up by 4 ($n \to n+4$)

$$= \sum_{i=5}^{n+4} (i-4)^2 = (5-4)^2 + (6-4)^2 + (7-4)^2 + \ldots + (n+4-4)^2$$

down by 4 ($i \to i-4$)

up by 4 ($1 \to 5$)

$$= \sum_{i=-2}^{n-3} (i+3)^2 = (-2+3)^2 + (-1+3)^2 + \ldots + (n-3+3)^2$$

$$= 1^2 + 2^2 + \ldots + n^2$$

"What goes up must come down" (Compensation!)

(If i in term goes up/down, endpoints go down/up by the same amount)

In general – no one unique way to write __any__ sum.

__Example__ $\displaystyle\sum_{i=1}^{5} 3i = 3.1 + 3.2 + 3.3 + 3.4 + 3.5$

$$= 3.(1+2+3+4+5)$$

$$= 3 \sum_{i=1}^{5} i$$

In general, we refer to general terms of a sum as a_i or b_i (like a function taking input i, where i must be an integer)

So in general (for any integers $m \le n$)
$$\sum_{i=m}^{n} c a_i = c \sum_{i=m}^{n} a_i$$
<u>if</u> c is constant relative to i

Similarly
$$\sum_{i=m}^{n} a_i \pm \sum_{i=m}^{n} b_i = (a_m + a_{m+1} + a_{m+2} + \ldots + a_n) \pm (b_m + b_{m+1} + b_{m+2} + \ldots + b_n)$$

$$(a_m \pm b_m) + (a_{m+1} \pm b_{m+1}) + \ldots + (a_n \pm b_n)$$

$$= \sum_{i=m}^{n} (a_i \pm b_i)$$

<u>Formulas</u> Want to reduce to these if possible.

(a) $$\sum_{i=1}^{n} 1 = \underset{i=1}{1} + \underset{i=2}{1} + \underset{i=3}{1} + \ldots + \underset{i=n}{1} = n$$ (= # terms)

$$\sum_{i=m}^{n} 1 \to = \underset{i=m}{1} + \ldots + \underset{i=n}{1} = n - (m-1)$$

$\to$ or use
$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{m-1} a_i + \sum_{i=m}^{n} a_i$$
This is a VERY USEFUL FACT in general; corresponds to rebracketing:
$$= (a_1 + a_2 + \ldots + a_{m-1}) + (a_m + \ldots + a_n)$$

For more on this, see discussion at the end of the file with solutions to the exercises.

(b) $2\sum_{i=1}^{n} i = 1+2+3+\ldots+(n-2)+(n-1)+n$ } This is the sum

$+\ (n+(n-1)+(n-2)+\ldots+3+2+1)$ } This is also the sum (written backwards)

Add the two versions of the sum together to get twice the sum

$= (n+1)+(n+1)+(n+1)+\ldots+(n+1)+(n+1)+(n+1)$

$= n(n+1)$ → If twice the sum is $n(n+1)$, then the sum is

So $\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$.

(c) $\boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$

(d) $\boxed{\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2}$

Exerise What is $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{5}{n}\left(\left(\frac{i+1}{n}\right)^2 + \frac{2i}{n}\right)$?

→ Work the sum – break it down into simpler pieces using the tricks given & use formulas to evaluate the simpler pieces.

Telescoping Sums → collapse

Example Find $\sum_{i=1}^{53}\left(\frac{1}{i} - \frac{1}{i+1}\right)$.

→ Does not fit in with the nice formulas above.

Solution

$\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \cancel{\left(\frac{1}{2} - \frac{1}{3}\right)} + \cancel{\left(\frac{1}{3} - \frac{1}{4}\right)} \cancel{+\ldots+}$

$\cancel{\left(\frac{1}{51} - \frac{1}{52}\right)} + \cancel{\left(\frac{1}{52} - \frac{1}{53}\right)} + \left(\cancel{\frac{1}{53}} - \frac{1}{54}\right)$

If in doubt, WRITE IT OUT!!!

$$= 1 - \frac{1}{54} = \underline{\underline{\frac{53}{54}}}.$$

Exercise Find $\displaystyle\sum_{i=1}^{10} \left(i^2 - (i+2)^2\right)$.

How many ways can you think up to solve this?