1A03 – CALCULUS I FOR SCIENCE
(SECTION C02)

Exercises from Lecture 22

Exercise What is $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{5}{n}\left(\left(\frac{i+1}{n}\right)^2 + \frac{2i}{n}\right)$?

Solution First focus on the sum ↑ :

$$\sum_{i=1}^{n} \frac{5}{n}\left(\left(\frac{i+1}{n}\right)^2 + \frac{2i}{n}\right) = \frac{5}{n}\sum_{i=1}^{n}\left(\left(\frac{i+1}{n}\right)^2 + \frac{2i}{n}\right)$$

pull out (does not depend on i)

split into 2 sums

$$= \frac{5}{n}\left(\sum_{i=1}^{n}\left(\frac{i+1}{n}\right)^2 + \sum_{i=1}^{n}\frac{2i}{n}\right)$$

pull out the denominators as these do not depend on i

$$= \frac{5}{n}\left(\frac{1}{n^2}\sum_{i=1}^{n}(i+1)^2 + \frac{1}{n}\sum_{i=1}^{n}2i\right)$$

We know how to deal with $\sum i^2$ so do an index shift to change this to that (changing endpoints appropriately)

pull out the 2 as it does not depend on i

↳ what goes down: $i+1 \xrightarrow{-1} i$, must go up: $\sum_{i=1}^{n} \xrightarrow{+1} \sum_{i=2}^{n+1}$:

$$= \frac{5}{n}\left(\frac{1}{n^2}\sum_{i=2}^{n+1} i^2 + \frac{2}{n}\sum_{i=1}^{n} i\right)$$

Alas, we have a formula for $\sum_{i=1}^{n+1} i^2$, not $\sum_{i=2}^{n+1} i^2$.

But what's the difference?

$$\sum_{i=1}^{n+1} i^2 = 1^2 + 2^2 + 3^2 + \ldots + (n+1)^2$$

$$\& \sum_{i=2}^{n+1} i^2 = (\;)\, 2^2 + 3^2 + \ldots + (n+1)^2$$

So this sum is 1 less than this sum

$$\text{i.e. } \sum_{i=2}^{n+1} i^2 = \sum_{i=1}^{n+1} i^2 - 1$$

$$= \frac{5}{n}\left(\frac{1}{n^2}\left(\sum_{i=1}^{n+1} i^2 - 1\right) + \frac{2}{n}\sum_{i=1}^{n} i\right)$$

We have formulas for these! Just pay attention to endpoints.

n+1 not n

$$= \frac{5}{n}\left(\frac{1}{n^2}\left(\frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} - 1\right) + \cancel{\frac{2}{n}}\left(\frac{\cancel{n}(n+1)}{\cancel{2}}\right)\right)$$

simplify this

cancellation

$$= \frac{5}{n}\left(\frac{1}{n^2}\left(\frac{(n+1)(n+2)(2n+3)}{6} - 1\right) + n+1\right)$$

$$= \frac{5}{n}\left(\frac{1}{n^2}\left(\frac{(n^2+3n+2)(2n+3)}{6} - 1\right) + n+1\right)$$

$$= \frac{5}{n}\left(\frac{1}{n^2}\left(\frac{2n^3+9n^2+13n+6}{6} - 1\right) + n+1\right)$$

Expand $= \dfrac{5(2n^3+9n^2+13n+6)}{6n^3} - \dfrac{5}{n^3} + 5 + \dfrac{5}{n}$

That is as good an expression as any for the sum in terms of n. Because what we really want to do is take the limit as n tends to ∞, we just need the expression for the sum to be something that lets us see that (& this will let us already):

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{5}{n}\left(\left(\frac{i+1}{n}\right)^2 + \frac{2i}{n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{5(2n^3+9n^2+13n+6)}{6n^3} - \frac{5}{n^3} + 5 + \frac{5}{n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{10 + \overset{\to 0}{\frac{45n^2}{n^3}} + \overset{\to 0}{\frac{65n}{n^3}} + \overset{\to 0}{\frac{30}{n^3}}}{6}\right) - \underset{\to 0}{\lim_{n\to\infty}\left(\frac{5}{n^3}\right)} + 5 + \underset{\to 0}{\lim_{n\to\infty}\left(\frac{5}{n}\right)}$$

$$= \frac{10}{6} + 5$$

$$= \underline{\underline{\frac{20}{3}}}$$

Exercise Find $\sum_{i=1}^{10}\left(i^2-(i+2)^2\right)$.

Solution Two solutions!

Solution 1 In the same spirit as above:

$$\sum_{i=1}^{10}\left(i^2-(i+2)^2\right)=\sum_{i=1}^{10} i^2-\sum_{i=1}^{10}(i+2)^2$$

Split into difference of 2 sums

2 choices here:
(A) Index shift
(B) Multiply out

Solution 1A : Index shift

$$=\sum_{i=1}^{10} i^2-\sum_{i=3}^{12} i^2$$

$$=\sum_{i=1}^{10} i^2-\left(\sum_{i=1}^{12} i^2-\sum_{i=1}^{2} i^2\right)$$

NOTICE $\sum_{i=1}^{n} a_i=\sum_{i=1}^{m} a_i+\sum_{i=m+1}^{n} a_i$ for any a_i, any integers $m,n \geq 1$.

(& similar trick works when i starts at the bottom at something other than $i=1$)

$$=\frac{10(11)(21)}{6}-\left(\frac{12(13)(25)}{6}-\frac{2(3)(5)}{6}\right)$$

$$=385-(650-5)$$

$$=\underline{\underline{-260}}.$$

Solution 1B : Multiply out

$$= \sum_{i=1}^{10} i^2 - \sum_{i=1}^{10} (i^2 + 4i + 4)$$

$$= \sum_{i=1}^{10} i^2 - \left(\sum_{i=1}^{10} i^2 + 4\sum_{i=1}^{10} i + 4\sum_{i=1}^{10} 1 \right)$$

$$= \cancel{\sum_{i=1}^{10} i^2} - \cancel{\sum_{i=1}^{10} i^2} - 4\sum_{i=1}^{10} i - 4\sum_{i=1}^{10} 1$$

$$= -\frac{4(10)(11)}{2} - 4(10 - (1-1))$$

$$= -220 - 4(10)$$

$$= -220 - 40 = \underline{\underline{-260}}.$$

Solution 2 Telescoping: write it out!

$$\sum_{i=1}^{10} \left(i^2 - (i+2)^2\right) = (1^2 - \cancel{3^2}) + (2^2 - \cancel{4^2}) + (\cancel{3^2} - \cancel{5^2}) + (\cancel{4^2} - \cancel{6^2})$$
$$+ \cancel{5^2} - \cancel{7^2}) + (\cancel{6^2} - \cancel{8^2}) + (\cancel{7^2} - \cancel{9^2})$$
$$(\cancel{8^2} - \cancel{10^2}) + (\cancel{9^2} - 11^2) + (\cancel{10^2} - 12^2)$$

4 terms remain; 2 of first type (from i^2) at beginning,
2 of second type (from $(i+2)^2$) at end.

$$= 1^2 + 2^2 - 11^2 - 12^2 = 1 + 4 - 121 - 144 = -260.$$

Remark We said in class that $\sum_{i=m}^{n} 1 = n-(m-1)$.

There are (at least) 2 ways to think about this.

First, think about how many terms there are in the sum.
Each term is 1, so the sum = # terms.

Count them off:
$$\underset{m\text{th}}{1} + \underset{(m+1)\text{st}}{1} + \underset{(m+2)\text{nd}}{1} + \cdots + \underset{(n-1)\text{st}}{1} + \underset{n\text{th}}{1}$$

If you're not used to thinking in these general terms, give yourself some pairs of integers m, n & check:

$$\sum_{i=3}^{7} 1 = \underset{i=3}{1} + \underset{i=4}{1} + \underset{i=5}{1} + \underset{i=6}{1} + \underset{i=7}{1} = 5 = 7-(3-1) = 7-2.$$

$$\sum_{i=1}^{2} 1 = \underset{i=1}{1} + \underset{i=2}{1} = 2 = 2-(1-1) = 2-0.$$

$$\sum_{i=-3}^{2} 1 = \underset{i=-3}{1} + \underset{i=-2}{1} + \underset{i=-1}{1} + \underset{i=0}{1} + \underset{i=1}{1} + \underset{i=2}{1} = 6 = 2-(-3-1)$$
$$= 2-(-4)$$
$$= 2+4.$$

You can also think of it like this, and this is an important trick to remember in lots of situations (it featured in solutions to both exercises above): in general $\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i$.

The second sum starts at $i=m+1$ because you don't want to count the a_m term twice:

$$\sum_{i=1}^{n} a_i = (a_1 + a_2 + \ldots + a_{m-1} + a_m) + (a_{m+1} + a_{m+2} + \ldots + a_{n-1} + a_n)$$
$$= \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i .$$

That means in particular, in our special case of $\sum_{i=m}^{n} 1$, we know

$$\sum_{i=1}^{n} 1 = \underbrace{1+\ldots+1}_{n \text{ times}} = n$$

(and $\sum_{i=1}^{m-1} 1 = m-1$ by the same reasoning)

and

$$\sum_{i=1}^{n} 1 = \sum_{i=1}^{m-1} 1 + \underbrace{\sum_{i=m}^{n} 1}$$

This is the sum we're interested in.

i.e.

$$n = (m-1) + \sum_{i=m}^{n} 1$$

So $\sum_{i=m}^{n} 1 = n - (m-1)$.

All very algebraic, so again plug in some values to check what's going on:

$$\sum_{i=1}^{6} 1 = \sum_{i=1}^{2} 1 + \sum_{i=3}^{6} 1$$

$$6 = 2 + \sum_{i=3}^{6} 1 \quad \text{so} \quad \sum_{i=3}^{6} 1 = 6-2 = 4$$

$(= 6-(3-1))$.