1A03 – Calculus I for Science

(Section C02)

Lecture 23

Last time <u>Sigma Notation</u>

$$\sum_{i=m}^{n} a_i = \text{"the sum from } i=m \text{ to } n \text{ of } a_i\text{"}$$

Like a function of i where i takes integer inputs

<u>Useful</u>:

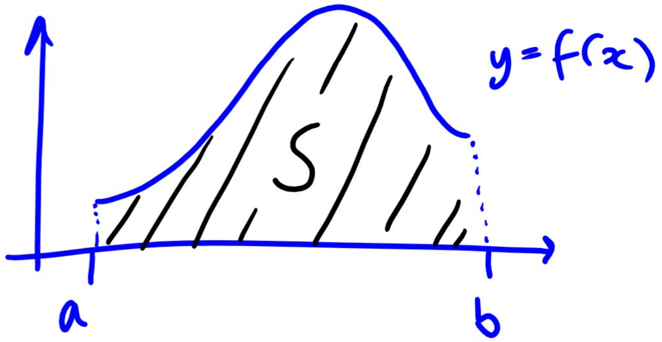
$$\sum_{i=1}^{n} 1 = n \; ; \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \; ; \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \; ; \quad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

5.1 Areas (and Distances) ← $v(t)\cdot t$

Read up on the "<u>Distance Problem</u>" in the textbook — it's basically the Area Problem in a not very good disguise.

<u>Area Problem</u>

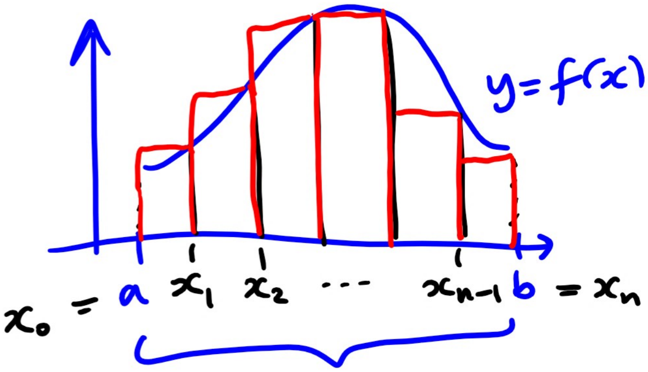
Given a positive continuous function $f(x)$, find the area under $y=f(x)$ between $x=a$ and $x=b$.



S = region under $y=f(x)$

<u>Goal</u>: Find A = area of S.

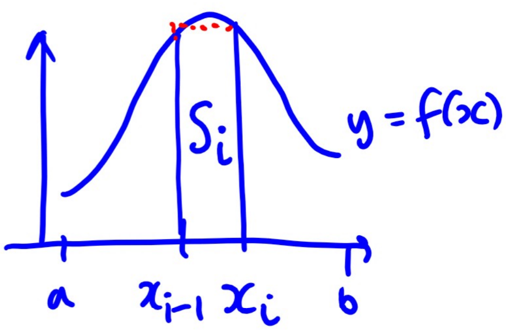
<u>Big Idea</u>: Cut up S into thin vertical slices and approximate each slice by a rectangle.



Sum of areas of rectangles $\approx$ area of S

Divide up $[a,b]$ into n mini-intervals of equal width $\Delta x = \dfrac{b-a}{n}$

$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$



Approximate S_i (ith slice)
= slice under $y=f(x)$ between $x = x_{i-1}$ and $x = x_i$

with rectangle of width Δx and height $f(x_i)$.

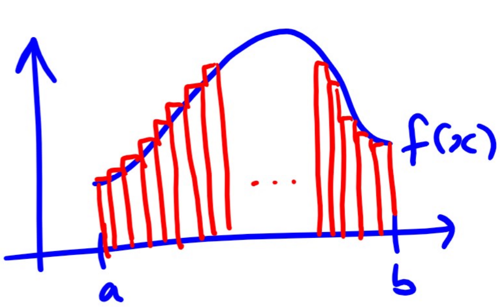
The area of $S_i \cong \Delta x \cdot f(x_i)$

This just means "We will now define R_n to be ..."

The area $A \cong R_n := \Delta x \cdot f(x_1) + \Delta x f(x_2) + \Delta x \cdot f(x_3) + \ldots + \Delta x \cdot f(x_n)$

"right endpoints"

$$= \sum_{i=1}^{n} \Delta x f(x_i)$$



As we increase # slices ($= n$) we get better & better approx. to A.

We take this to the limit and <u>define</u> the area A under $y = f(x)$ from $x = a$ to $x = b$ to be

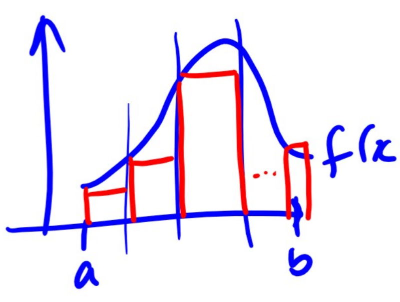
$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta x \, f(x_i) .$$

<u>Note</u>: It can be shown that this limit must exist as $f(x)$ is continuous.

<u>Note</u>: No good reason to choose right endpoints to get height of our rectangles; could choose any point in each $[x_{i-1}, x_i]$.

e.g. could choose left endpoint x_{i-1} & make rectangle approx. S_i have height $f(x_{i-1})$

then area of $S_i \cong \Delta x . f(x_{i-1})$



We're defining L_n here to be RHS

$$\text{So } A \cong L_n := \sum_{i=1}^{n} \Delta x . f(x_{i-1})$$

Since $f(x)$ is continuous

$$A = \lim_{n \to \infty} L_n \quad \text{(as well)}$$

Or we could take <u>any</u> so-called <u>sample point</u> x_i^* in $[x_{i-1}, x_i]$ – very common choice is midpoint.

i.e. $\frac{x_i + x_{i-1}}{2}$

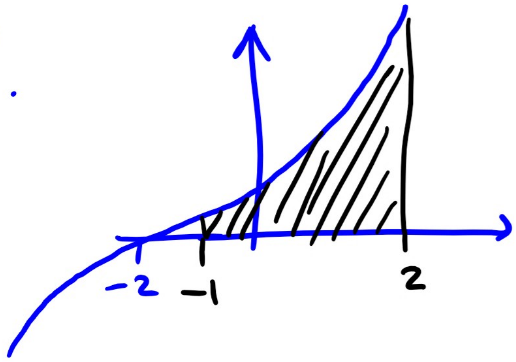
Now : area of $S_i \cong \Delta x \,.\, f(x_i^*)$

So $A \cong \underbrace{\sum_{i=1}^{n} \Delta x \,.\, f(x_i^*)}_{\text{Riemann sum}}$

($f(x)$ continuous

— $\lim_{n\to\infty}$ (each Riemann sum)

is same & $= A$.)



Example Let $f(x) = (x+2)^3$.

Estimate A, the area under $y = f(x)$ on the interval $[-1, 2]$, using six rectangles and midpoints.

Solution First divide $[\overset{a}{\overset{\shortparallel}{-1}}, \overset{b}{\overset{\shortparallel}{2}}]$ into $\overset{n}{\overset{\shortparallel}{6}}$ subintervals of length $\Delta x = \dfrac{b-a}{n} = \dfrac{2-(-1)}{6} = \dfrac{1}{2}$.

So our intervals are $[-1, -\frac{1}{2}]$, $[-\frac{1}{2}, 0]$, $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$, $[1, \frac{3}{2}]$, $[\frac{3}{2}, 2]$.

TBC.