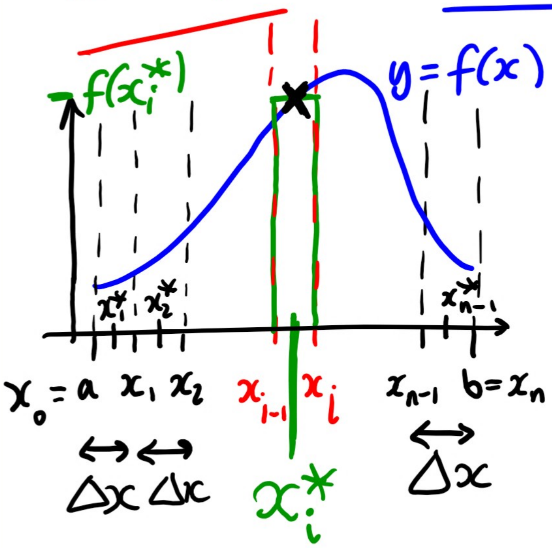
1A03 — Calculus I for Science

(Section C02) Lecture 24

Last time: The Area Problem & Riemann Sums



A = area under $y=f(x)$

$$= \lim_{n\to\infty} \boxed{\sum_{i=1}^{n} \Delta x\, f(x_i^*)}$$

where x_i^* is a sample point in $[x_{i-1}, x_i]$

and $\Delta x = \frac{b-a}{n}$.

Example $f(x) = (x+2)^3$

Approx. A, area under $y=f(x)$, by six $\overset{=n}{}$ rectangles & midpoints $\hookrightarrow x_i^*$ on $[\underset{a}{-1}, \underset{b}{2}]$.

Solution $\Delta x = \frac{b-a}{n} = \frac{2-(-1)}{6} = \frac{1}{2}$.

So intervals are $[-1,-\frac{1}{2}], [-\frac{1}{2},0], [0,\frac{1}{2}], [\frac{1}{2},1], [1,\frac{3}{2}], [\frac{3}{2},2]$.

Sample points: $\underset{=x_1^*}{-\frac{3}{4}}, \underset{=x_2^*}{-\frac{1}{4}}, \underset{=x_3^*}{\frac{1}{4}}, \underset{=x_4^*}{\frac{3}{4}}, \underset{=x_5^*}{\frac{5}{4}}, \underset{=x_6^*}{\frac{7}{4}}$

Heights of corresponding rectangles: $f(-\frac{3}{4}), f(-\frac{1}{4}), f(\frac{1}{4}), f(\frac{3}{4}), f(\frac{5}{4}), f(\frac{7}{4})$

So we have $A \cong \sum_{i=1}^{6} \Delta x\, f(x_i^*)$

$$= \frac{1}{2}\left(\frac{5}{4}\right)^3 + \frac{1}{2}\left(\frac{7}{4}\right)^3 + \frac{1}{2}\left(\frac{9}{4}\right)^3 + \frac{1}{2}\left(\frac{11}{4}\right)^3 + \frac{1}{2}\left(\frac{13}{4}\right)^3 + \frac{1}{2}\left(\frac{15}{4}\right)^3$$

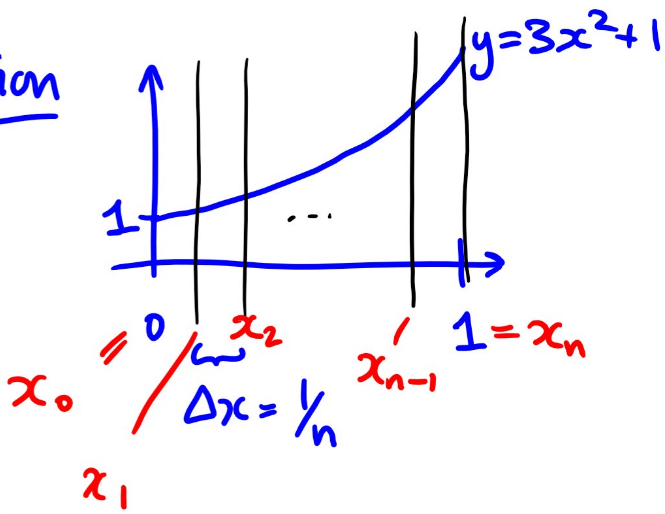
$$= \underbrace{\frac{1}{2}\left(\frac{1}{4^3}\right)}_{1/128}\Big(\underbrace{5^3+7^3+9^3+11^3+13^3+15^3}_{8100}\Big)$$

$$= \frac{8100}{128} \cong \underline{\underline{63.28}}.$$

__Example__ Let $f(x) = 3x^2+1$ on $[0,1]$.

Find A area under $y = f(x)$.

__Solution__



$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

Right endpoint:

$$x_i = 0 + \frac{i}{n}$$

Choose as the ith sample point

$$x_i^* = x_i \text{ for each } i$$

sample point

Useful to remember:

$x_0 = a$

$x_1 = a + \Delta x$

$x_2 = (a+\Delta x) + \Delta x = a + 2\Delta x$

$x_3 = a + 3\Delta x$

$$\boxed{x_i = a + i\Delta x}$$

Heights of rectangles: $f(x_i^*)$

$$= 3(x_i^*)^2 + 1$$

$$= 3\left(\frac{i}{n}\right)^2 + 1$$

Riemann sum : $\displaystyle\sum_{i=1}^{n} \Delta x\, f(x_i^*) = \sum_{i=1}^{n} \frac{1}{n}\left(3\left(\frac{i}{n}\right)^2 + 1\right)$

Now look at $\displaystyle\lim_{n\to\infty}\left(\sum_{i=1}^{n} \frac{1}{n}\left(3\left(\frac{i}{n}\right)^2+1\right)\right)$

$$= \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{3i^2}{n^2}+1\right)\right) = \lim_{n\to\infty}\left(\frac{1}{n}\left[\sum_{i=1}^{n}\frac{3i^2}{n^2} + \sum_{i=1}^{n} 1\right]\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}\left[\frac{3}{n^2}\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} 1\right]\right)$$

$$= \lim_{n\to\infty}\left(\frac{3}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{1}{n}(n)\right)$$

$$= \lim_{n\to\infty}\left(\frac{\cancel{3}(2n^3+3n^2+n)}{2\cancel{6}n^3} + 1\right) = \frac{2}{2} + 1 = \underline{\underline{2.}}$$

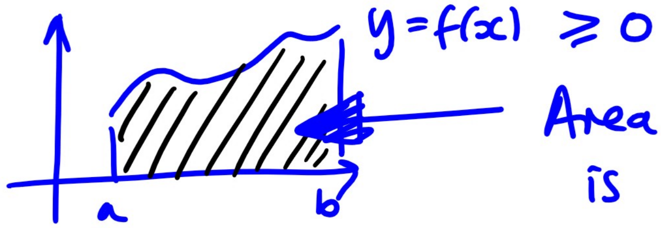
Think rational functions

$\dfrac{2n^3+3n^2+n}{2n^3} = \dfrac{2 + 3/n + 1/n^2}{2}$

5.2 The Definite Integral

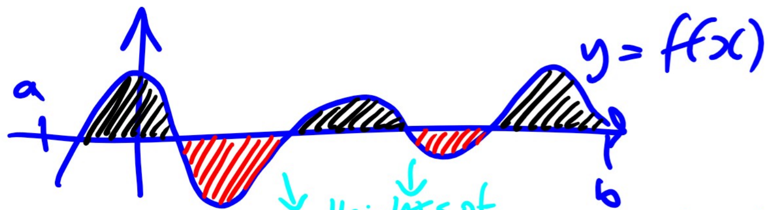
If $f(x)$ is defined on $[a,b]$, $\Delta x = \frac{b-a}{n}$, $[a,b]$ is subdivided into n subintervals of length Δx, & x_i^* is any sample point in the ith subinterval $[x_{i-1}, x_i]$, then the <u>definite integral</u> of $f(x)$ from a to b is the # $= \displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\Delta x\, f(x_i^*)$,

[Interpretation :



Area under $y=f(x)$ is $\int_a^b f(x)\,dx$.

If $f(x)$ is sometimes positive, Sometimes negative,



then $\int_a^b f(x)\,dx$ is

(area under $y=f(x)$ above x-axis)

$-$ (area above $y=f(x)$ below x-axis)

Heights of approximating rectangles $f(x_i^*) < 0$ so area of those rectangles is negative & gives negative contribution to Riemann sum

$=$ [black] $-$ [red]]

The above only makes sense IF limit exists (and agrees for all choices of sample points).

But if it does we say that $f(x)$ is <u>integrable</u>.

This <u>will</u> happen if $f(x)$ is continuous or has only finitely many jump discontinuities.

Properties of Definite Integral

(1) $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

$\Delta x = \dfrac{b-a}{n}$ as a factor everywhere.

T.B.C. ...