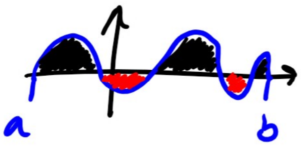
1A03 – CALCULUS I FOR SCIENCE
(SECTION C02)

Lecture 25

Last time: The DEFINITE Integral



$$\int_a^b f(x)\,dx = \text{(area under } y=f(x) \text{ above } x\text{-axis between } x=a \text{ and } x=b\text{)}$$
$$- \text{(area above } y=f(x) \text{ below } x\text{-axis between } x=a \text{ and } x=b\text{)}$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \, f(x_i^*)$$

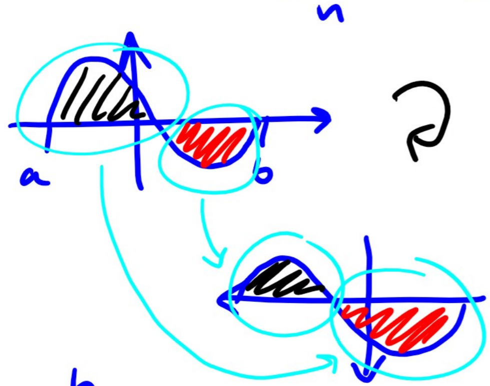
IF $f(x)$ integrable

i.e. IF limit exists & is same for all choices of sample points x_i^* e.g. $f(x)$ continuous

Properties of definite integral

① $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\Delta x = \frac{b-a}{n} \longrightarrow \frac{a-b}{n} = -\frac{(b-a)}{n}$$



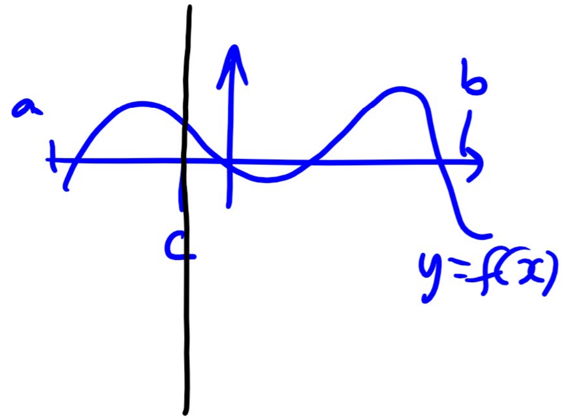
② $$\int_a^a f(x)\,dx = 0$$

③ $$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

④ $$\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$$ for any # k constant relative to x

[③ & ④ follow from similar properties of limits & Σ notation.]

⑤ $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$



⑥ If $f(x) \geq 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq 0$.

⑦ If $f(x) \geq g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

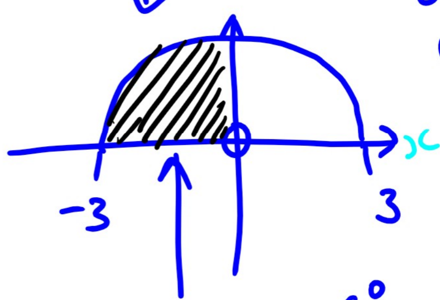
$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(s)\,ds = \int_a^b f(r)\,dr = \int_a^b f(y)\,dy$$

x is a "dummy variable" in the notation

Example Use properties of the definite integral to find $\int_{-3}^{0} \sqrt{9-x^2} - 1\,dx$.

Solution

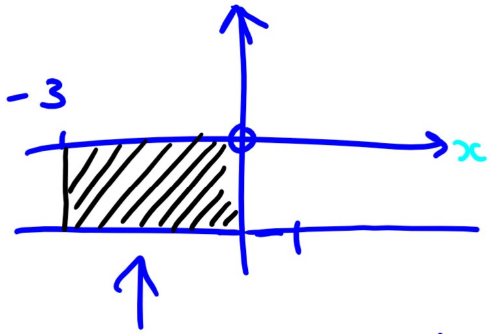
$$= \int_{-3}^{0} \sqrt{9-x^2}\,dx + \int_{-3}^{0} -1\,dx$$



$y = \sqrt{9 - x^2}$

$y^2 = 9 + x^2$

$x^2 + y^2 = 9$



$$\text{Area} = \int_{-3}^{0} \sqrt{9-x^2}\, dx = \frac{1}{4} \cdot \pi \cdot 3^2 = \frac{9}{4}\pi.$$

$$\text{Area} = 3 \times -1 = -3$$

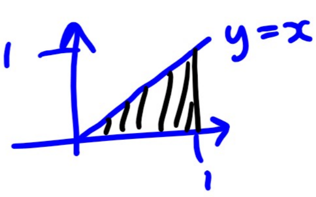
(or think "$3 \times 1 = 3$ <u>below</u> x-axis therefore negative)

So our answer: $$\int_{-3}^{0} \sqrt{9-x^2} - 1\, dx = \underline{\frac{9}{4}\pi - 3}.$$

Example Evaluate $\int_0^1 3x^2 + x + 1\, dx$.

Solution Last time with Riemann sum example we worked out $$\int_0^1 3x^2 + 1\, dx = 2$$ in disguise.

So $$\int_0^1 3x^2 + x + 1\, dx = \int_0^1 3x^2 + 1\, dx + \int_0^1 x\, dx$$

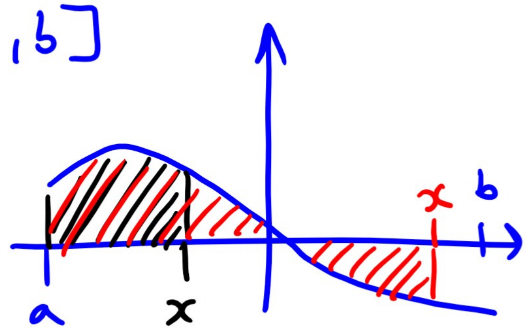


$$= 2 + \frac{1}{2} = 5/2.$$

5.3 The Fundamental Theorem of Calculus

t $f(t)$ be defined on $[a,b]$

Co ider the "area up to # x"

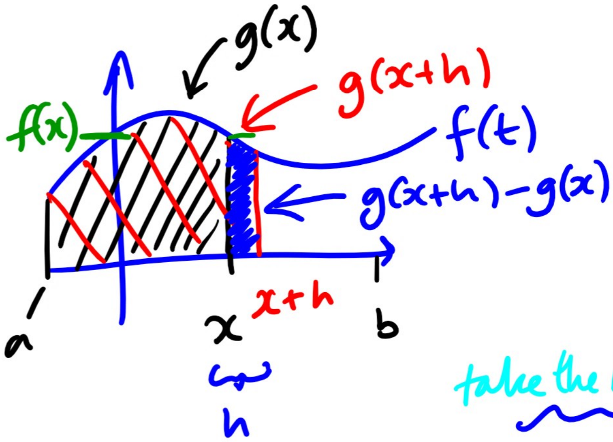


$$\int_a^c -t)\,dt$$ ← is a function of x (changes with x).

What i the derivative of this function

$$g(x) = \int_a^x f(t)\,dt\,?$$

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \quad \text{(if limit exists)}$$



$$g(x+h) - g(x) \approx h \cdot f(x)$$

$$\frac{g(x+h) - g(x)}{h} \approx f(x)$$

$$\xrightarrow[\text{it turns out}]{\text{take the limit as } h\to 0} g'(x) = f(x).$$

Fundamental Theorem of Calculus, Part I

If $f(x)$ is continuous on $[a,b]$, then

$g(x) = \int_a^x f(t)\,dt$ is also continuous on $[a,b]$, differentiable on (a,b) and $g'(x) = f(x)$.

Example Let $g(x) = \int_2^x t \sin(3t)\, dt$. Find $g'(x)$.

Solution let $f(t) = t \sin(3t)$.

So $g'(x) = f(x) = x \sin(3x)$ by F.T.C.

Example Let $h(x) = \int_5^{x^3} \frac{ds}{s^2}$. Find $\frac{dh}{dx}$.

$\frac{1}{s^2}\, ds$

Solution If we had

$$g(u) = \int_5^u \frac{ds}{s^2},$$

$f(s) = \frac{1}{s^2}$

then F.T.C. tells us $g'(u) = f(u) = \frac{1}{u^2}$.

So $h(x) = g(u(x))$ where $u(x) = x^3$

$$\frac{dh}{dx} = \frac{dg}{du} \cdot \frac{du}{dx} = \frac{1}{u^2} \cdot 3x^2 = \frac{1}{(x^3)^2} \cdot 3x^2 = \frac{3}{x^4}.$$

In general, if $h(x) = \int_a^{k(x)} f(t)\, dt$, then, by the Chain Rule,

$$h'(x) = f(k(x)) \cdot k'(x).$$