1A03 – CALCULUS I FOR SCIENCE
(SECTION C02)

Lecture 26

Last time FUNDAMENTAL THEOREM of CALCULUS

PART I Differentiation reverses integration
(Finding slopes of tangent lines) (finding area)

If $f(x)$ is continuous, $g(x) = \int_a^x f(t)\,dt$ has $g'(x) = f(x)$.

In other words $g(x)$ is an antiderivative of $f(x)$.

Suppose we have any antideivative of $f(x)$.
↳ call it $F(x)$

We know $F(x) = g(x) + C$.

So consider net change of $F(x)$ over $[a,b]$:

$$F(b) - F(a) = \left(g(b) + \cancel{C}\right) - \left(g(a) + \cancel{C}\right)$$
$$= g(b) - g(a) \quad \leftarrow \int_a^a f(t)\,dt = 0$$
$$= \int_a^b f(t)\,dt.$$

F.T.C. Part II If $f(x)$ continuous on $[a,b]$
then $\int_a^b f(t)dt = F(b) - F(a)$

for <u>any</u> choice of antiderivative $F(x)$ of $f(x)$.

We often write as $\Big[F(x)\Big]_a^b$ or $F(x)\Big]_a^b$ or $F(x)\Big|_a^b$

<u>Example</u> Find $\displaystyle\int_0^1 3x^2+1 \, dx$.

(Notice – we already found the answer in an earlier exercise involving Riemann sums)

<u>Solution</u>
$$\int_0^1 \underbrace{3x^2+1}_{f(x)} \, dx = \Big[\underbrace{x^3 + x}_{F(x)}\Big]_0^1$$

$F(x)=x^3+x$ is <u>AN</u> antiderivative of $f(x)=3x^2+1$.

$$= (1^3+1) - (0^3+0)$$
$$= 2.$$

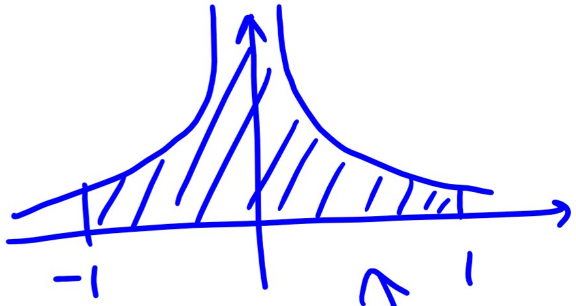
<u>Example</u> Find $\displaystyle\int_5^{10} \frac{1}{x} - e^x \, dx$

<u>Solution</u>
$$\int_5^{10} \underbrace{\frac{1}{x} - e^x}_{f(x)} \, dx = \Big[\underbrace{\ln|x| - e^x}_{F(x)}\Big]_5^{10}$$
$$= (\ln 10 - e^{10}) - (\ln 5 - e^5)$$
$$= \#.$$

<u>Example</u> Find $\displaystyle\int_{-1}^1 \frac{1}{x^4} \, dx$.

$$\frac{1}{x^4} = x^{-4} \longrightarrow \frac{x^{-3}}{-3}$$

<u>Solution</u> Isn't this $= \Big[-\dfrac{1}{3x^3}\Big]_{-1}^1$



$$= -\frac{1}{3(1)^3} - \left(-\frac{1}{3(-1)^3}\right) = -\frac{1}{3} - \frac{1}{3}$$

$$= -\frac{2}{3}.\ ?$$

NO

+ve PROBLEM -ve

F.T.C. only applies when $f(x)$ is continuous.

So above WAS NOT VALID.

Actual answer: $\int_{-1}^{1} \frac{1}{x^4}\,dx$ DNE.

How about $\int_0^2 xe^{x^2}\,dx$?

Can't use so-far known rules of antiderivatives

5.5 Substitution Rule

First goal is find $F(x)$ with $F'(x) = xe^{x^2}$.

Strategy To simplify a complicated integrand :

($f(x)$ in $\int f(x)\,dx$)

① Find a part of the integrand whose derivative appears as a factor in the integrand (up to a constant multiple)

(2) Call this part found in (1) u.
Then rewrite <u>everything</u> in terms of u.

Now to integrate:

(3) Integrate (4) Rewrite everything back in terms of original variable

---

<u>Example</u> Find $\int x e^{x^2} dx$.

<u>Solution</u> (1) Notice $\frac{d}{dx}(x^2) = 2x$ which is a constant multiple of factor x in xe^{x^2}.

(2) So set $u = x^2$ Then we have $\frac{du}{dx} = 2x$

i.e. $x = \frac{1}{2}\frac{du}{dx}$.

So $\int x e^{x^2} dx = \int \frac{1}{2} \frac{du}{\cancel{dx}} e^u \cancel{dx}$

The MAGIC is we allow ourselves to "cancel" the dx as though it were a #

$= \int \frac{1}{2} e^u \, du$

move to end

substitute x back in

(3) Integrate $= \frac{1}{2}e^u + C$ (4) $= \frac{1}{2}e^{x^2} + C$.

CHECK: $\frac{d}{dx}\left(\frac{1}{2}e^{x^2}+C\right) = \frac{1}{2}\cdot(2x)e^{x^2} = \underline{\underline{xe^{x^2}}}.$

Chain Rule

This process "undoes"/"reverses" Chain Rule.

Justification – for the strategy above

If $f(x)$ has antiderivative $F(x)$, $(F'(x) = f(x))$

then $\frac{d}{dx}\left[F(g(x))\right] = F'(g(x)).g'(x) = f(g(x)).g'(x).$

i.e. $\int f(g(x))g'(x)\,dx = F(g(x)) + C$

Set $u = g(x)$ $\frac{du}{dx} = g'(x)$

$$\int f(u)\left(\frac{du}{dx}\right)dx = F(u) + C$$
$$= \int f(u)\,du$$

Must "be" same so "cancellation" works.

To summarize:

Substitution Rule If $u = g(x)$, then

$$\int f(g(x)).g'(x)\,dx = \int f(u)\,du$$

Example Find $\int s^4\sqrt{3s^5-1}\,ds.$

our goal: identify $u=g(x)$ in the integrand.

Solution $u = s^5$. (or $u = 3s^5 - 1$).

↳ Try it: can get answer with only one run through our strategy!

$$\frac{du}{ds} = 5s^4$$

(!) $du = 5s^4\,ds$

$ds = du/5s^4$

Now that we are allowed to "cancel", we can pretend to work with du & ds here as though they were #s in order to set up the new integral correctly.

$$= \int \cancel{s^4}\sqrt{3u-1}\,\frac{du}{5\cancel{s^4}}$$

$$= \underbrace{\frac{1}{5}\int \sqrt{3u-1}\,du}$$

Run strategy again! Notice $v = 3u$ or $v = u$ doesn't help!

$v = 3u - 1$

$$\frac{dv}{du} = 3$$

$$du = \frac{dv}{3}$$

$$= \frac{1}{5}\int \sqrt{v}\,\frac{dv}{3} = \frac{1}{15}\int \sqrt{v}\,dv$$

$$= \frac{1}{15}\left(\frac{2}{3}\right)v^{3/2} + C$$

$\sqrt{v} = v^{1/2}$

$\to v^{3/2}\left(\frac{2}{3}\right)$

$$= \frac{2}{45}v^{3/2} + C$$

Now reverse substitutions:

in terms of u $= \frac{2}{45}(3u-1)^{3/2} + C$

in terms of s $= \frac{2}{45}(3s^5-1)^{3/2} + C$.

How to find $\int_0^2 xe^{x^2}\,dx$?

2 methods:

① Use F.T.C. Part II (we found $\int xe^{x^2}dx = \frac{1}{2}e^{x^2}$ above)

$$= \left[\frac{1}{2}e^{x^2}\right]_0^2$$

$$= \frac{1}{2}e^4 - \frac{1}{2}e^0$$

$$= \frac{1}{2}(e^4 - 1).$$

(2) Definite Integral Version of the Substitution Rule

If $u = g(x)$, then
$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

So in our example we get (recall: $u = x^2 = g(x)$)

$$\int_0^2 x e^{x^2}\,dx = \int_{0^2=0}^{2^2=4} \frac{1}{2} e^u\,du$$

$$= \int_0^4 \frac{1}{2} e^u\,du = \left[\frac{1}{2}e^u\right]_0^4$$

$$= \frac{1}{2}e^4 - \frac{1}{2}e^0$$

$$= \frac{1}{2}(e^4 - 1).$$

as before.

We'll do some more examples of this next time & justify this version of the rule, but otherwise this is all the material you need for the 2nd Midterm Test!!!