# 1A03 – Calculus I for Science
(Section C02)

Lecture 27

## Last time: Substitution Rule

If $u = g(x)$, then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Values over which $x$ ranges. (limits $a$, $b$)

Values over which $u = g(x)$ ranges. (limits $g(a)$, $g(b)$)

**Justification** If $F(x)$ is an antiderivative of $f(x)$

then $F(g(x))$ " " " $f(g(x))g'(x)$

so $\displaystyle\int_a^b f(g(x))\,g'(x)\,dx = \Big[F(g(x))\Big]_a^b = F(g(b)) - F(g(a))$ FTC II

$\displaystyle = \Big[F(u)\Big]_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f(u)\,du$

**Example** Find $\displaystyle\int_1^2 \frac{\sin(\pi/x)}{x^2}\,dx$.

**Solution** Substitute $u = \dfrac{\pi}{x}$. $\dfrac{du}{dx} = -\dfrac{\pi}{x^2}$ so think

$$dx = \frac{du}{(-\pi/x^2)} = -\frac{x^2\,du}{\pi}$$

$$\int_1^2 \frac{\sin(\pi/x)}{x^2}\,dx = \int_{\pi/1}^{\pi/2} \frac{\sin(u)}{\cancel{x^2}}\left(-\frac{\cancel{x^2}\,du}{\pi}\right)$$

$$= \int_{\pi}^{\pi/2} -\frac{\sin(u)}{\pi}\,du$$

Could also do:

$$\frac{1}{\pi}\int_{\pi}^{\pi/2} -\sin(u)\,du$$

$$= \frac{1}{\pi}\Big[\cos(u)\Big]_{\pi}^{\pi/2}$$

$$= \frac{1}{\pi}\left(\cos\left(\frac{\pi}{2}\right) - \cos(\pi)\right)$$

$$= \frac{1}{\pi}\left(0 - (-1)\right) = \frac{1}{\pi}.$$

$$= \frac{1}{\pi}\int_{\pi/2}^{\pi} \sin(u)\,du = \frac{1}{\pi}\Big[-\cos(u)\Big]_{\pi/2}^{\pi}$$

$$= \frac{1}{\pi}\left(-\cos(\pi) - \left(-\cos\left(\pi/2\right)\right)\right)$$

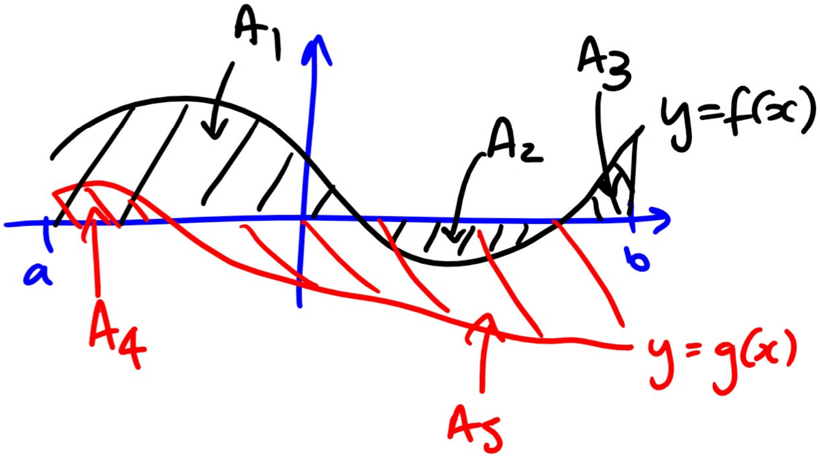
$$= \frac{1}{\pi}\left(-(-1) - 0\right) = \underline{\underline{\frac{1}{\pi}}}.$$

UP TO HERE FOR TEST #2

# APPLICATIONS OF INTEGRATION

## 6.1 Areas between curves

2 functions on $[a, b]$ $\quad f(x) \geq g(x)$




$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3$$

$$\int_a^b g(x)\,dx = A_4 - A_5$$
$$= -(-A_4 + A_5)$$

$$\text{Area between } f(x) \text{ \& } g(x) = \underline{A_1} - A_4 + A_5 \underline{-A_2} \underline{+A_3}$$

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

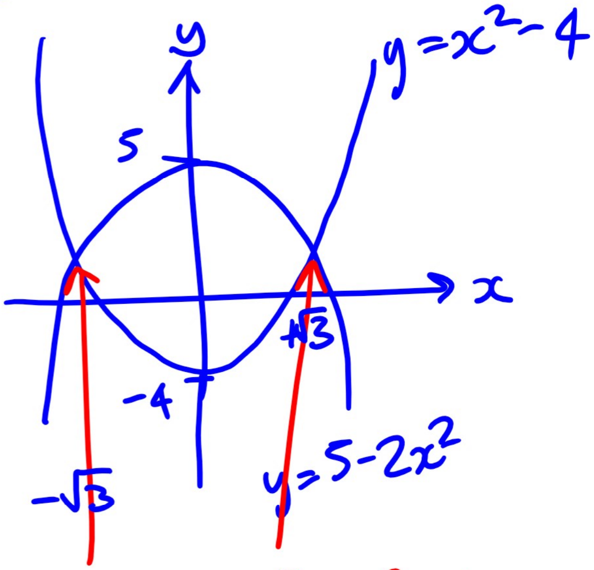
$$= \int_a^b f(x) - g(x)\,dx$$

In general if $f(x) \geq g(x)$ on $[a, b]$,

the area of region bounded by $y = f(x)$ (above) and $y = g(x)$ (below) is $\displaystyle\int_a^b f(x) - g(x)\,dx$.

<u>Example</u> Find the area of the region enclosed by $y = 5 - 2x^2$ and $y = x^2 - 4$.

<u>Solution</u> Setup: find $f(x) \geq g(x)$ and $[a, b]$.




Solve $5 - 2x^2 = x^2 - 4$

$x = \pm\sqrt{3}$

$f(x) = 5 - 2x^2$ top

$g(x) = x^2 - 4$ bottom

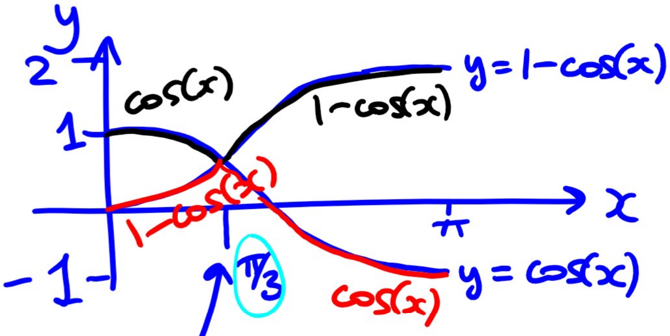
$$\text{Area} = \int_{-\sqrt{3}}^{\sqrt{3}} (5 - 2x^2) - (x^2 - 4)\,dx$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} 9 - 3x^2\,dx = \Big[9x - x^3\Big]_{-\sqrt{3}}^{\sqrt{3}}$$

$$= \cdots = \underline{\underline{12\sqrt{3}}}.$$

<u>Example</u> Find the area bounded by $y = \cos(x)$ and $y = 1 - \cos(x)$ on $[0, \pi]$.

**Solution**




$$1-\cos(x) = \cos(x)$$
$$1 = 2\cos(x)$$
$$\cos(x) = 1/2$$
$$x = \pi/3$$

$$\int_0^{\pi/3} \underbrace{\cos(x) - (1-\cos(x))}_{2\cos(x)-1}\,dx + \int_{\pi/3}^{\pi} \underbrace{1-\cos(x) - \cos(x)}_{1-2\cos(x)}\,dx$$

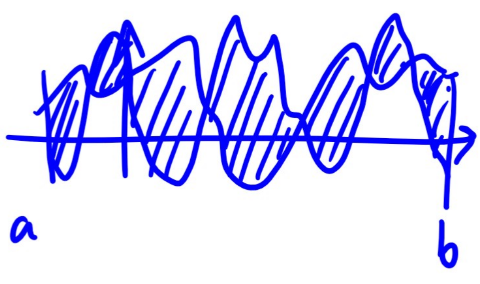
Notice: $2\cos(x)-1 \geq 0$ on $[0, \frac{\pi}{3}]$

$\leq 0$ on $[\pi/3, \pi]$

So in fact we're integrating

$|2\cos(x)-1|$ everywhere

In general if we have $f(x)$, $g(x)$ which cross, area between them is $\displaystyle\int_a^b |f(x)-g(x)|\,dx$

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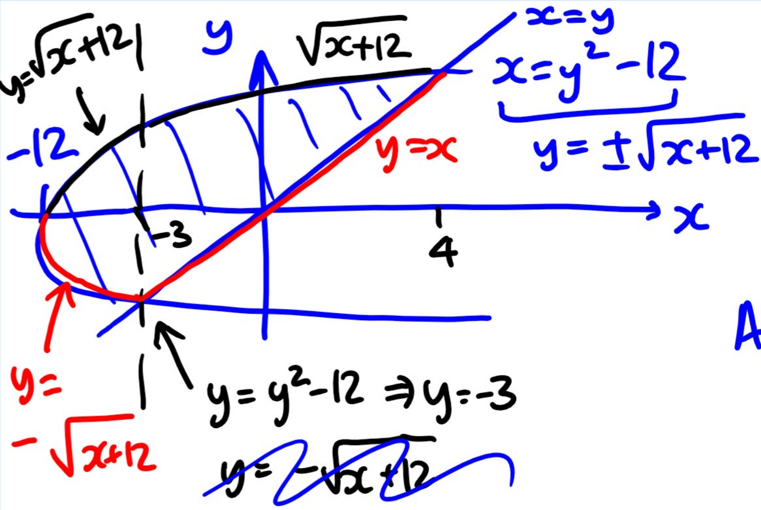
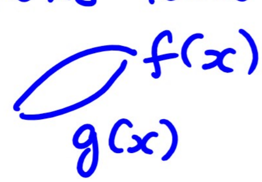
**Example** Find the area of the region enclosed by $x = y^2 - 12$ and $x = y$.

**Solution**



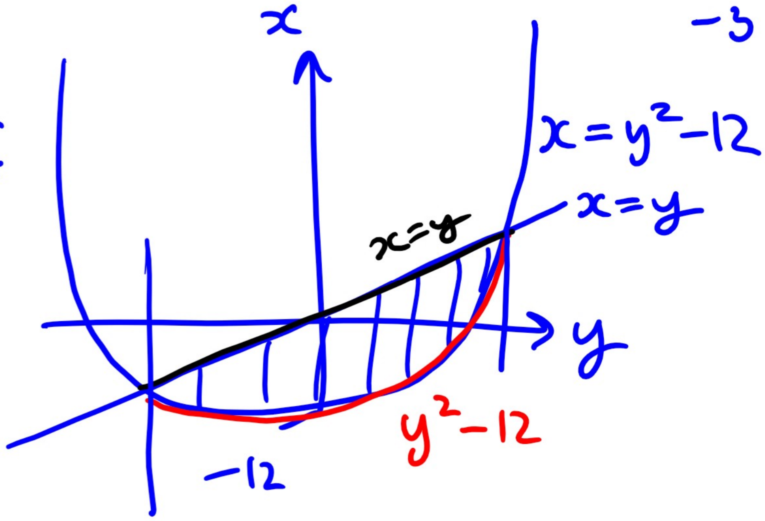

Want 2 regions that look like




$$\text{Area} = \int_{-12}^{-3} \sqrt{x+12} - \left(-\sqrt{x+12}\right) dx$$

$$+ \int_{-3}^{4} \sqrt{x+12} - x \; dx$$

<u>**OR**</u>




<u>Exercise</u>