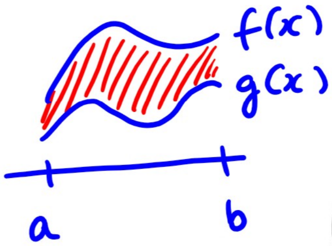
# 1A03 – CALCULUS I FOR SCIENCE
(SECTION C02) — Lecture 28

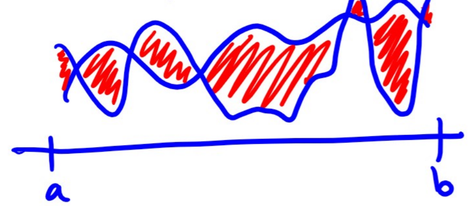
## Last time: AREAS BETWEEN CURVES

If $f(x) \geq g(x)$ on $[a,b]$, then the area of the region between $y=f(x)$ and $y=g(x)$ is given by

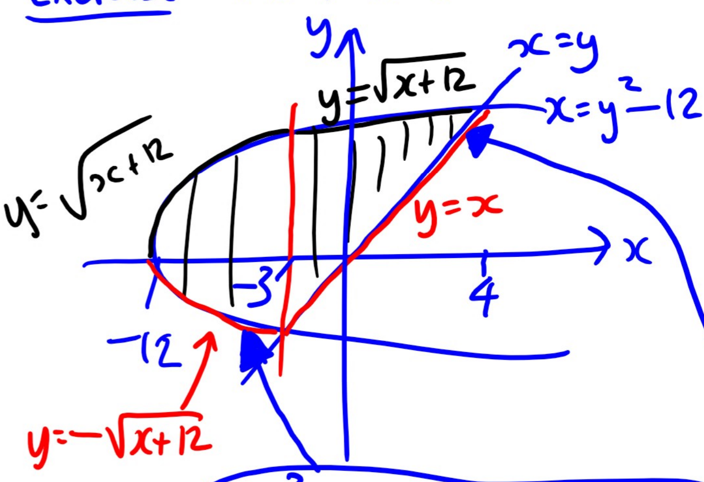



$$\boxed{\int_a^b \underbrace{f(x) - g(x)}\, dx.}$$

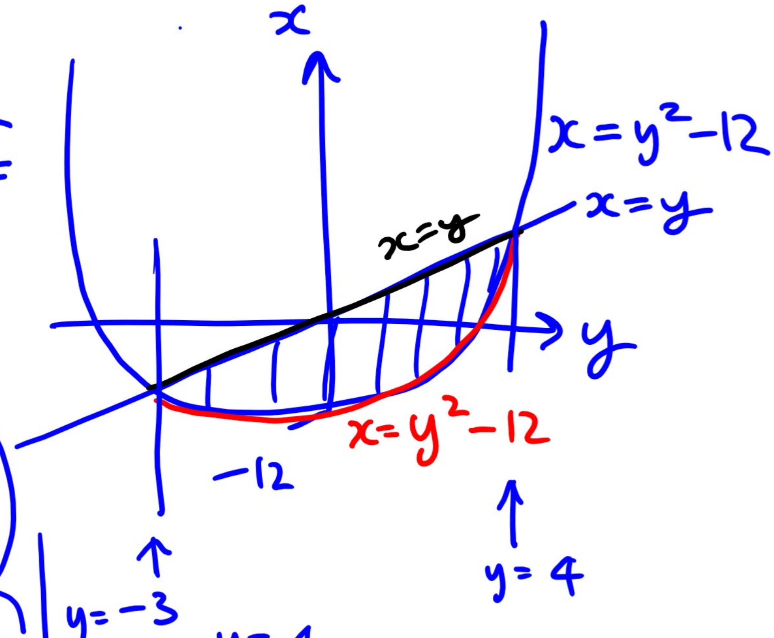
If $f(x)$ & $g(x)$ cross, this is $|f(x) - g(x)|$.




**Exercise** Find area bounded by $x = y^2 - 12$ and $x = y$.




OR $=$




$$\text{Area} = \int_{-12}^{-3} \sqrt{x+12} - (-\sqrt{x+12})\, dx + \int_{-3}^{4} \sqrt{x+12} - x\, dx$$

2 messy integrals!

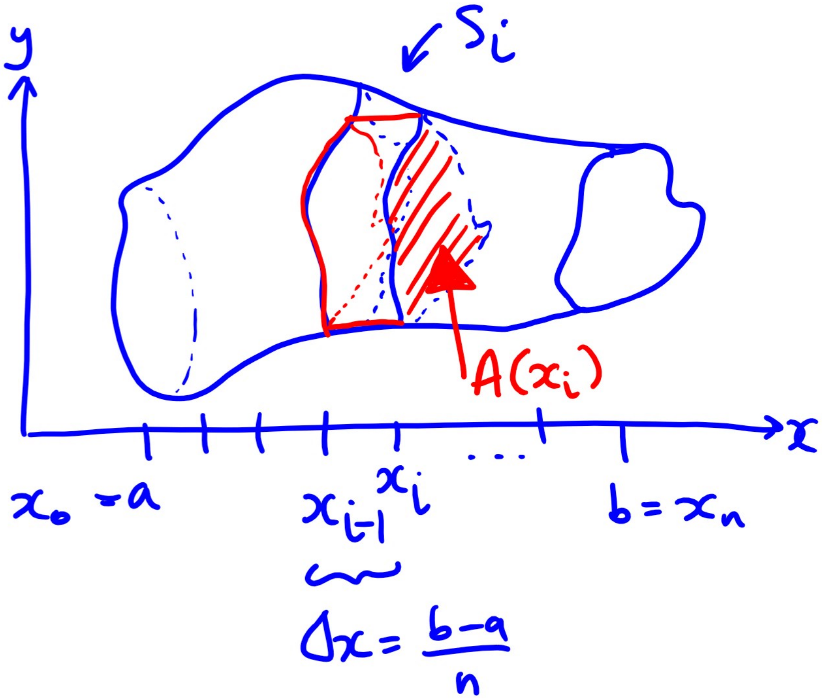
$$\text{Area} = \int_{y=-3}^{y=4} y - (y^2 - 12)\, dy$$

Much nicer integral!

$$= \left[ \frac{y^2}{2} - \frac{y^3}{3} + 12y \right]_{-3}^{4} = \dots = \frac{343}{6}.$$

## 6.2 Volumes

3D analog to Area Problem.




To approximate the volume of this solid:

- Slice into $n$ pieces vertically, called $S_i$ at evenly spaced $x$-values $x_0 = a, x_1, \dots, x_{n-1}, x_n = b$ of width $\Delta x = \frac{b-a}{n}$.

- Approximate volume of ith slice $S_i$ by volume of straight-sided shape with "width" $\Delta x$ & "side" area $A(x_i)$ i.e. volume of $S_i \approx A(x_i) \cdot \Delta x$

  ↑ cross-sectional area at $x_i$

- Total volume $\approx \sum_{i=1}^{n} A(x_i) \cdot \Delta x$

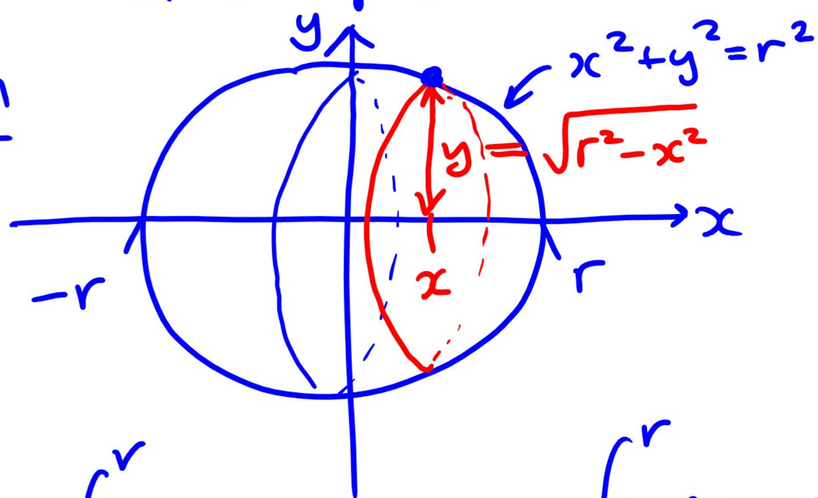
- Actual volume $= \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i) \cdot \Delta x$

$$= \int_a^b A(x)\,dx$$

So challenge here is, given a solid, finding cross-sectional area $A(x)$ in terms of $x$.

<u>Example</u> Use above method to find volume of sphere of radius $r$.

<u>Solution</u>



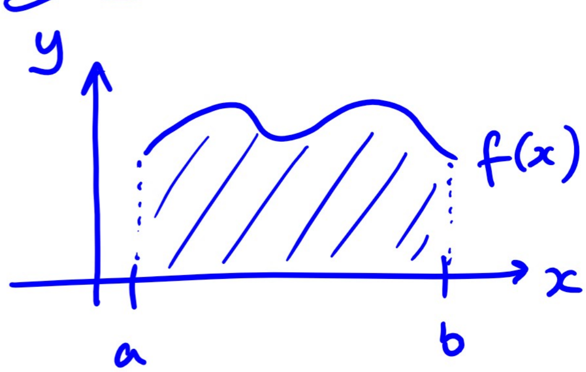

$$A(x) = \pi y^2 = \pi\left(\sqrt{r^2-x^2}\right)^2 = \pi(r^2-x^2)$$

$$\text{Volume} = \int_{-r}^{r} \pi(r^2-x^2)\,dx = 2\int_0^r \pi(r^2-x^2)\,dx = 2\pi\left[r^2x - \frac{x^3}{3}\right]_0^r$$

$$= 2\pi\left(r^3 - \frac{r^3}{3} - 0\right)$$

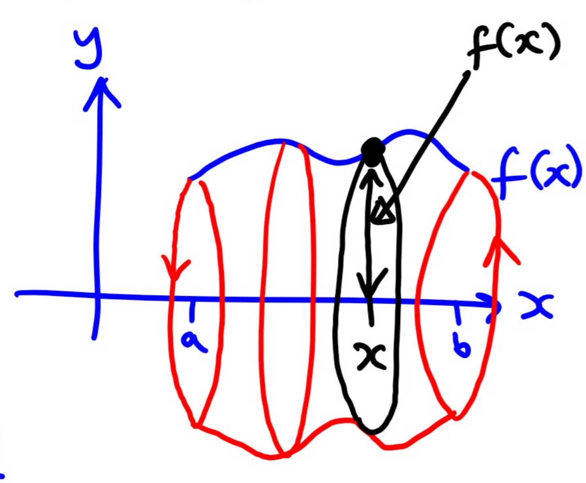
$$= \underline{\underline{\frac{4\pi r^3}{3}}}.$$

## <u>Solids of Rotation</u> – solids of a special form:




$\longrightarrow$

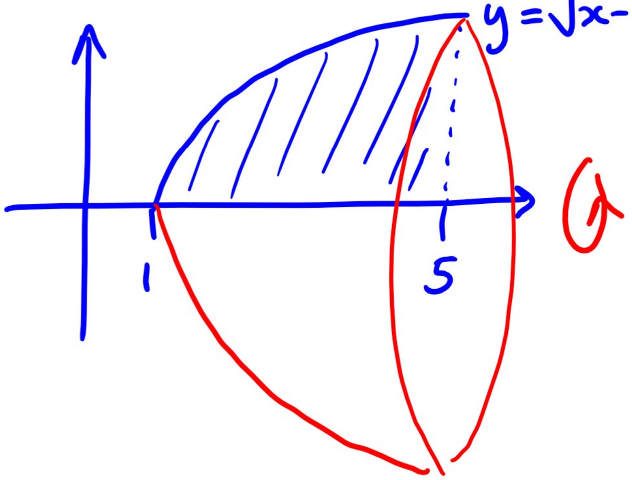
Rotate $y=f(x)$ curve around $x$-axis to make a solid



$A(x) = \pi(f(x))^2$ So volume $= \displaystyle\int_a^b \pi(f(x))^2\,dx$.

**Example** Find volume of solid generated by rotating $y = \sqrt{x-1}$ from $x=1$ to $x=5$ about $x$-axis.

**Solution**




$$\text{volume} = \int_1^5 \pi\left(\sqrt{x-1}\right)^2 dx$$

$$= \int_1^5 \pi(x-1)\,dx$$

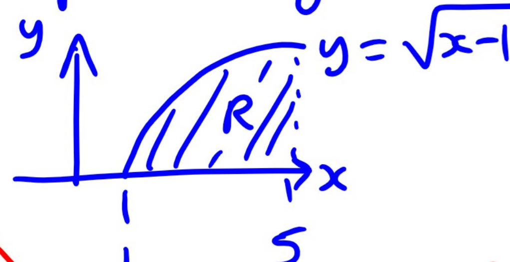
$$= \pi\left[\frac{x^2}{2} - x\right]_1^5$$

$$= \pi\left(\frac{25}{2} - 5 - \left(\frac{1}{2} - 1\right)\right)$$

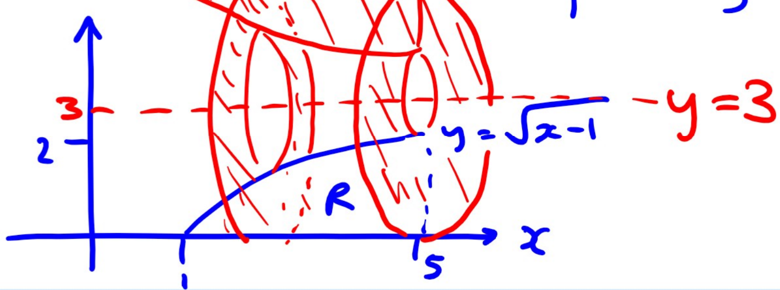
$$= \underline{\underline{8\pi}}.$$

Can use this idea to understand more complicated shapes:

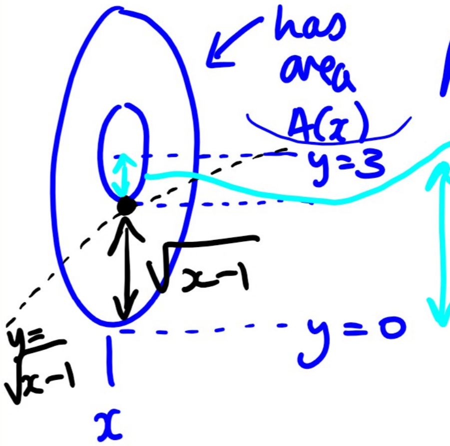
**Example** Find volume of solid generated by rotating $R$ about $y = 3$.




**Solution**




Cross-section is washer-shaped $\bigcirc$




Area $A(x)$ = area of outer disc − area of inner disc

$$= \pi . 3^2 - \pi(3-\sqrt{x-1})^2$$

$$= (6\sqrt{x-1} - x + 1)\pi$$

So volume $= \displaystyle\int_1^5 (6\sqrt{x-1} - x + 1)\pi \, dx \ldots$

$$= \underline{\underline{24\pi}}.$$