1A03 – CALCULUS I FOR SCIENCE
(SECTION C02)

Lecture 30

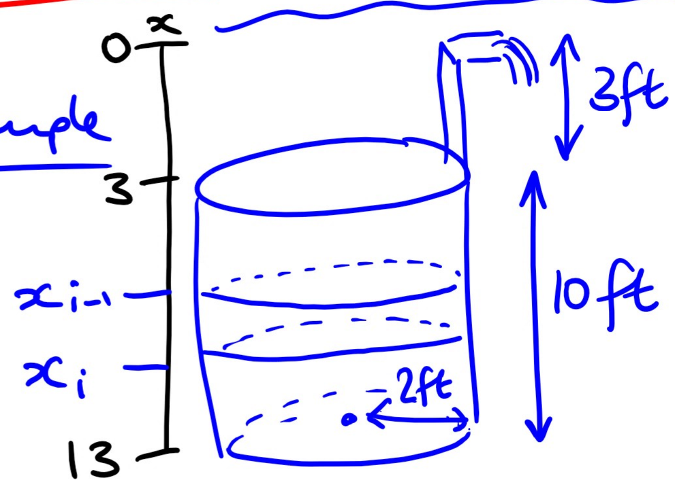
Last time WORK = FORCE × DISTANCE

To move an object along a straight line from $x=a$ to $x=b$ where the force needed at x is given as $f(x)$:

Measured in Joules = Nm or ftlb

WORK DONE : $W = \int_a^b f(x)\,dx$.

Example



? Work needed to pump all the water in the tank up to the spout?

- Cut up $[3,13]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \frac{13-3}{n} = \frac{10}{n}$.

  → get cylinders of water

- Volume of ith cylinder at $[x_{i-1}, x_i] = \Delta x . \pi . 2^2$
  $= 4\pi \Delta x$.

- W_i = work required to move the ith cylinder
  = Force × distance

$\approx$ (Volume $\times$ water weight) $\times\, x_i^*$ for a sample point in $[x_{i-1}, x_i]$

$= 4\pi\Delta x \times 62.5\ \text{lb}/\text{ft}^3 \times x_i^*$ ← given this

$= 250\pi x_i^* \Delta x.$

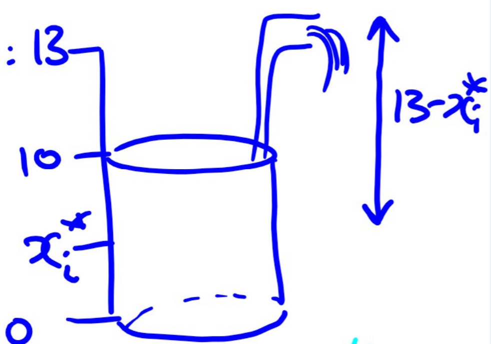
- Total work $\approx \displaystyle\sum_{i=1}^{n} \underbrace{250\pi x_i^*}\,\underbrace{\Delta x}.$

- Total work $= \displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} 250\pi x_i^*\Delta x = \int_3^{13} 250\pi x\,dx$

$$= \Big[125\pi x^2\Big]_3^{13} = 125\pi(169-9)$$

$$= \underline{\underline{20{,}000\pi\ \text{ftlbs}}}$$

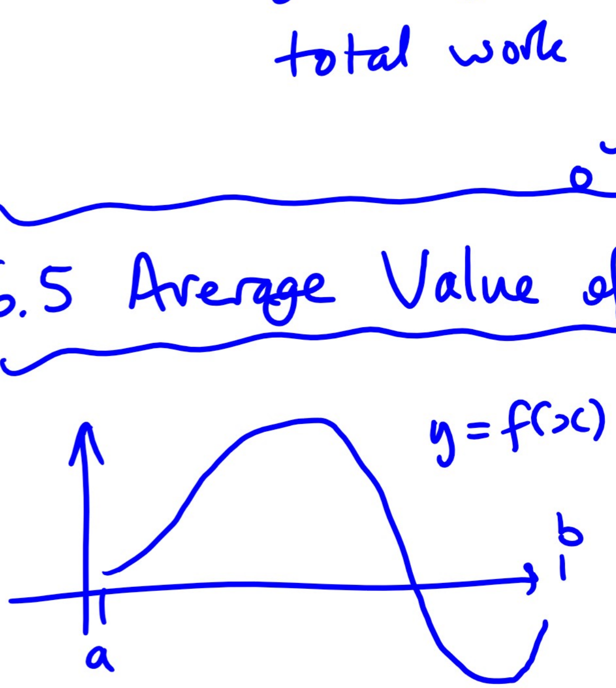
<u>Notice</u> You could put the x-axis like this →



& you'd get total work $\displaystyle\int_0^{10} 250\pi(13-x)\,dx$.

(Check it gives the same answer!)

6.5 Average Value of a Function



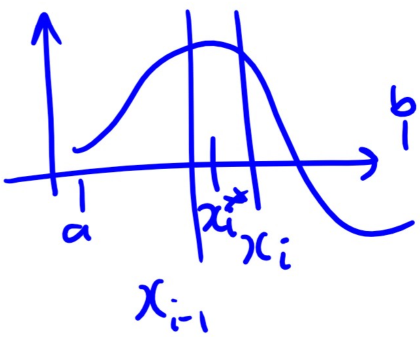
Suppose $f(x)$ is temp. at at time x.

How to find average temp. f_{ave}?

Have to take an average over all possible $f(x)$ for all possible $x \in [a,b]$!

Temp. reading every hour $\longrightarrow$ can average the readings & get an approximation

Every minute $\rightarrow$ better approx.
Every second $\longrightarrow$ even better approx.



- Divide up $[a,b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \frac{b-a}{n}$.

- For each i find $f(x_i^*)$ for a sample point $x_i^* \in [x_{i-1}, x_i]$.

- Average these values: $\dfrac{\sum_{i=1}^{n} f(x_i^*)}{n} = \dfrac{\sum_{i=1}^{n} f(x_i^*)}{(b-a/\Delta x)}$

Notice $n = \dfrac{b-a}{\Delta x}$

$$= \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*) \Delta x$$

- We define $f_{ave} = \dfrac{1}{b-a} \left(\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x \right)$

$$= \frac{1}{b-a} \int_a^b f(x)\, dx .$$

Example An alpine trail has elevation (in m) given by $h(x) = x^5 - 20x^3 + 350$ (x in km). What is the average elevation over the first 5 km?

Solution

$$h_{ave} = \frac{1}{5-0}\int_0^5 x^5 - 20x^3 + 350 \, dx$$

$$= \frac{1}{5}\left[\frac{x^6}{6} - 5x^4 + 350x\right]_0^5$$

$$= \ldots = \underline{\underline{245\tfrac{5}{6} \text{ m}}}.$$

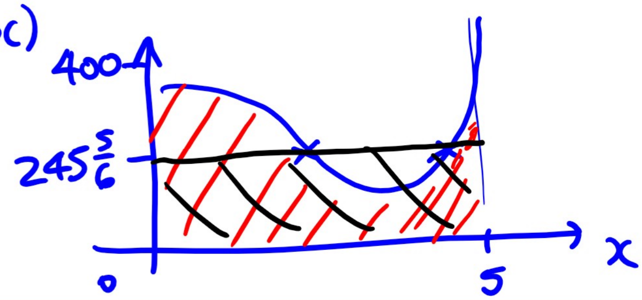
Follow-up Q: Is there a point on trail at which we stand at elevation $h_{ave} = 245\frac{5}{6}$ m?

A: Yes as $h(x)$ is continuous:

M.V.T. for Integrals If $f(x)$ is continuous on $[a,b]$, then <u>there exists</u> $c \in [a,b]$ with $f(c) = f_{ave}$.

$$= \frac{1}{b-a}\int_a^b f(x)\,dx \quad \text{i.e. } (b-a)f(c) = \int_a^b f(x)\,dx.$$

Look at $h(x)$



$$\int_0^5 h(x)\,dx = \underbrace{h(c)}_{h_{ave}}\cdot(5-0)$$

7.1 Integration by Parts

– sort of reverses Product Rule

Product Rule:

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x)$$

$$f(x)g(x) = \int f(x)g'(x) + f'(x)g(x)\,dx$$

$$= \int f(x)g'(x)\,dx + \int f'(x)g(x)\,dx$$

So Integration by Parts formula:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

This ↗ is not so memorable. Easier to remember:

Use Subtitution Rule: $u = f(x)$, Then $\frac{du}{dx} = f'(x) \to du = f'(x)dx$

$v = g(x)$. $\frac{dv}{dx} = g'(x) \to dv = g'(x)dx$

$$\to \boxed{\int u\,dv = uv - \int v\,du}$$

Strategy for choosing u & dv:

(1) Let u be a factor that becomes "simpler" when we differentiate it.

(2) Let dv be a factor whose antiderivative we know.

<u>Example</u> Find $\int x^{1/3} \ln x \, dx$

<u>Solution</u> 2 choices for u & dv:

① $u = x^{1/3}$ $\quad \frac{du}{dx} = \frac{1}{3} x^{-2/3}$

$dv = \ln x$ $\quad v = ???$

② $u = \ln x$ $\quad \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x^{1/3}$ $\quad v = \left(\frac{3}{4}\right) x^{4/3}$

So $\int x^{1/3} \ln x \, dx = \frac{3}{4} x^{4/3} \ln x - \int \frac{3}{4} x^{1/3} \, dx$.

$= \frac{3}{4} x^{4/3} \ln x - \left(\frac{3}{4}\right)^2 x^{4/3} + C$.