1A03 – CALCULUS I FOR SCIENCE

(SECTION C02) Lecture 31

Last time INTEGRATION BY PARTS

Rearranging the PRODUCT RULE gives: $\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$

Which is easier to remember as: $\int u\,dv = uv - \int v\,du$

u : gets simpler when you differentiate it!

e.g. $u = x^2, x^3, \ldots$ or $\ln x$ or $\arcsin(x), \arctan(x)$.

dv : you know how to integrate it!

Examples

Reminder: (from last time)

$$\int \underbrace{x^{1/3}}_{dv} \underbrace{\ln x}_{u}\,dx = \frac{3}{4}x^{4/3}\ln x - \int \frac{3}{4}x^{4/3-1}\,dx$$

$$= \frac{3}{4}x^{4/3}\ln x - \left(\frac{3}{4}\right)^2 x^{4/3} + C.$$

Example Find $\int \underbrace{x^2}_{u} \underbrace{\cos(x)\,dx}_{dv}$.

Solution

$u = x^2 \quad \frac{du}{dx} = 2x$

$\frac{dv}{dx} = \cos(x) \quad v = \sin(x)$

$t = 2x \quad \frac{dt}{dx} = 2$

$\frac{ds}{dx} = \sin(x) \quad s = -\cos(x)$

$$= x^2\sin(x) - \int 2x\sin(x)\,dx$$

$$= x^2\sin(x) - \left(-2x\cos(x) - \int 2(-\cos(x))\,dx\right)$$

$$= x^2\sin(x) + 2x\cos(x) - 2\int \cos(x)\,dx$$

$$= x^2\sin x + 2x\cos x - 2\sin x + C$$

Example Find $\int x^3 \cos(x^2)\,dx$.

Solution Not obvious how to use Int. by Parts ?

Try substitution (seems promising) : try $t = x^2$

$$\frac{dt}{dx} = 2x \qquad dx = \frac{dt}{2x}$$

$$\int x^3 \cos(x^2)\,dx = \int \overset{t}{\cancel{x^{3}}} \cos(t)\,\frac{dt}{2\cancel{x}}$$

$$= \frac{1}{2}\int \underbrace{t}_{u}\,\underbrace{\cos(t)\,dt}_{dv}$$

$u = t \quad \frac{du}{dt} = 1$

$\frac{dv}{dt} = \cos(t) \quad v = \sin(t)$

$$= \frac{1}{2}\left(t\sin(t) - \int \sin(t)\,dt\right)$$

$$= \frac{1}{2}\left(t\sin(t) + \cos(t)\right) + C$$

Undo substitution

$$= \frac{1}{2}\left(x^2\sin(x^2) + \cos(x^2)\right) + C.$$

Remark : good choices for u : $x^2, x^3, \ldots$ $\ln x$

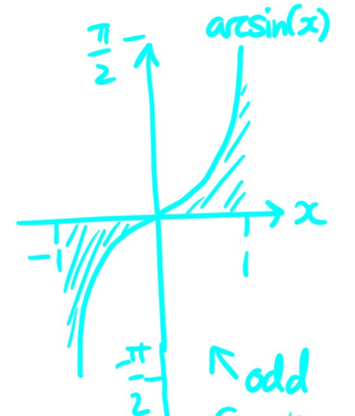
not so good choices for u : $\sin x, \cos x, e^x$.

Definite Integral Form:

$$\int_a^b f(x)\,g'(x)\,dx = \Big[f(x)\,g(x)\Big]_a^b - \int_a^b g'(x)\,f(x)\,dx$$

i.e. endpoints don't change

Example Find $\displaystyle\int_{-1}^{1} \underbrace{\sin^{-1}(x)}_{u}\,\underbrace{dx}_{dv}.$



odd function on a symmetric interval about 0 must have net area = 0.

Solution $\dfrac{du}{dx} = (\sin^{-1}(x))' = \dfrac{1}{\sqrt{1-x^2}}$ $\qquad \dfrac{dv}{dx} = 1$

$v = x$

$$\int_{-1}^{1} \sin^{-1}(x)\,dx = \Big[x\sin^{-1}(x)\Big]_{-1}^{1} - \int_{-1}^{1} \frac{x}{\sqrt{\underbrace{1-x^2}_{t}}}\,dx$$

$t = 1-x^2$

$\dfrac{dt}{dx} = -2x$

$dx = \dfrac{dt}{-2x}$

$$= \Big[x\sin^{-1}(x)\Big]_{-1}^{1} - \int_{1-(-1)^2=0}^{1-1^2=0} \frac{\cancel{x}}{\sqrt{t}}\,\frac{dt}{-2\cancel{x}} \longrightarrow 0$$

Nothing wrong here — we get $-\frac{1}{2}\displaystyle\int_0^0 \frac{1}{\sqrt{t}}\,dt$ which must be 0 as $\displaystyle\int_a^a f(x)\,dx = 0$ always.

$$= 1\cdot\frac{\pi}{2} - (-1)\left(-\frac{\pi}{2}\right) - 0$$

$$= \frac{\pi}{2} - \frac{\pi}{2} = 0$$

Notice: $\sin^{-1}(x)$ on $[-1, 1]$ is an odd function on a symmetric interval (about 0) so

$$\int_{-1}^{1} \sin^{-1}(x)\,dx = 0 \text{ by observation.}$$

(If all else fails — or maybe, better check first!!)

7.2 Trigonometric Integrals

Example Find $\int \sin x \cos^2 x \, dx$

Solution Substitute $u = \cos x$

$$\frac{du}{dx} = -\sin x \qquad dx = \frac{du}{-\sin x}$$

$$\rightarrow \int \cancel{\sin x}\, u^2 \frac{du}{-\cancel{\sin x}} = -\int u^2 du = -\frac{u^3}{3} + C$$

$$= -\frac{1}{3}\cos^3 x + C.$$

Example Find $\int \sin^3 x \, dx$.

Solution Want to do $u = \sin x$ but no $\frac{du}{dx}$ factor
$= \cos x$

Here we use trig. identity

$$\cos^2 x + \sin^2 x = 1$$

to get a mixture of $\sin x$, $\cos x$ terms:

$$\int \sin^3 x \, dx = \int \sin x \sin^2 x \, dx = \int \underbrace{\sin x (1 - \cos^2 x)}_{\sin x - \sin x \cos^2 x} dx$$

$$= \int \sin x \, dx - \int \sin x \cos^2 x \, dx$$

$$= -\cos x - \left(-\frac{1}{3}\cos^3 x\right) + C$$

$$= -\cos x + \frac{1}{3}\cos^3 x + C.$$

Similarly can handle any odd power of $\sin x$ or $\cos x$

Example Find $\int \cos^7 x \, dx$.

Solution

$$\int \cos x \, . \cos^6 x \, dx = \int \cos x \, (\cos^2 x)^3 \, dx$$

$$= \int \cos x \, (1 - \sin^2 x)^3 \, dx$$

$$= \int (1-u^2)^3 \, du$$

Now substitute: $u = \sin x$

$$\frac{du}{dx} = \cos x$$

$\longrightarrow$ can find this by substitution or just expand out & solve. (In this case, just expand; in general, maybe substitution will save you some work, but not here, alas.)