1A03 – CALCULUS I FOR SCIENCE
(SECTION C02) Lecture 32

Last time TRIGONOMETRIC INTEGRALS

e.g. $\int \cos^7 x \, dx$, $\int \sin^3 x \, dx$, $\int \sin x \cos^2 x \, dx$

If power is odd, split off one factor & use $\cos^2 x + \sin^2 x = 1$:

e.g. $\int \cos^7 x \, dx = \int \cos x (\cos^2 x)^3 dx = \int \cos x (1 - \sin^2 x)^3 dx$

Now substitute in u (where $\frac{du}{dx}$ = factor split off)

Example Find $\int \sin^7 x \cos^4 x \, dx$.

It doesn't matter what the power is of the other trig. function (the one that will be equal to u)

Solution Split off $\sin x$:

$$= \int \sin x \cdot \underbrace{\sin^6 x}_{\underbrace{(\sin^2 x)^3}_{(1-\cos^2 x)^3}} \cos^4 x \, dx$$

Now sub. $u = \cos x$

$\frac{du}{dx} = -\sin x$

$-du = \sin x \, dx$

$$= -\int (1-u^2)^3 u^4 \, du$$

→ Expand out & solve...

So to evaluate $\int \sin^n x \cos^m x \, dx$ and

– n odd, split off $\sin x$: $\sin x \cdot \underbrace{\sin^{n-1} x}\cos^m x$, replace $\sin^{n-1} x$ with

$(1-\cos^2 x)^{\frac{n-1}{2}}$ & substitute $u = \cos x$ | $\sin^{n-1}(x) = (\sin^2 x)^{\frac{n-1}{2}}$

- m odd, split $\cos x$: $\cos x \cdot \underbrace{\cos^{m-1}(x)} \cdot \sin^n x$

  replace $\cos^{m-1}(x)$ with $(1-\sin^2 x)^{\frac{m-1}{2}}$ & substitute $u = \sin x$.

- If both odd you choose!

- If both even : use $\frac{1}{2}$ angle formulae:

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1+\cos 2x).$$

<u>Example</u> Find $\int \cos^2 x \underbrace{\sin^4 x}_{(\sin^2 x)^2} \, dx$

<u>Solution</u>

$$= \int \left(\frac{1}{2}(1+\cos 2x)\right)\left(\frac{1}{2}(1-\cos 2x)\right)^2 dx$$

$$= \frac{1}{8}\int \underbrace{(1+\cos 2x)(1-\cos 2x)^2}_{\text{expand as a cubic in } \cos 2x} dx$$

$\vdots$

$$= \frac{1}{8}\int 1 - \cos 2x - \cos^2 2x + \cos^3 2x \, dx$$

$\downarrow u = 2x$

$$= \underbrace{\frac{1}{8}\int 1 - \cos 2x - \frac{1}{2}(1+\cos 4x)\, dx}_{\text{Straightforward substitution (or just do it)}} + \underbrace{\frac{1}{8}\cdot\frac{1}{2}\int \cos^3 u \, du}_{\text{Use above method.}}$$

— 3 is odd!

These ideas work with $\int \tan^n x \sec^m x \, dx$ —

here we split off factors either $\underbrace{\sec^2 x}_{=\frac{d}{dx}\tan x}$ or $\underbrace{\sec x \tan x}_{=\frac{d}{dx}\sec x}$

& use $\sec^2 x = 1 + \tan^2 x$.

<u>Example</u> Find $\int \tan^5 x \sec^3 x \, dx$.

<u>Solution</u> Try splitting off $\sec^2 x$: $\int \tan^5 x \underbrace{\sec x}_{\text{?? in terms of } \tan x \text{ ???}} \sec^2 x \, dx$

How about splitting off $\sec x \tan x$:

we get $\int \underbrace{\tan^4 x}_{\substack{=(\tan^2 x)^2 \\ =(\sec^2 x - 1)^2}} \sec^2 x \underbrace{\tan x \sec x \, dx}_{du \text{ if } u = \sec x}$

(So try to get rid of all the other $\tan x$ terms)

$$= \int (\sec^2 x - 1)^2 \sec^2 x \, (\tan x \sec x) \, dx$$

$$\underset{u = \sec x}{=} \int (u^2 - 1)^2 u^2 \, du \rightarrow \text{expand \& integrate.}$$

<u>Example</u> Find $\int \tan^2 x \sec^4 x \, dx$.

<u>Solution</u> Split off $\sec^2 x$: $\int \tan^2 x \underbrace{\sec^2 x}_{(1+\tan^2 x)} \underbrace{\sec^2 x \, dx}_{du \text{ if } u = \tan x}$

(so get rid of other $\sec x$ terms)

$$= \int \tan^2 x\,(1+\tan^2 x)\ \sec^2 x\,dx$$

$$\underset{u=\tan x}{=} \int u^2(1+u^2)\,du = \int u^2 + u^4\,du$$

$$= \frac{1}{3}u^3 + \frac{1}{5}u^5 + C$$

$$= \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C.$$

So to evaluate $\int \tan^n x \sec^m x\,dx$, and

- if m even, split off $\sec^2 x$, rewrite $\sec^{m-2} x$ as $(1+\tan^2 x)^{\frac{m-2}{2}}$ & substitute $u = \tan x$
- if n odd, split off $\sec x \tan x$, rewrite $\tan^{n-1} x$ as $(\sec^2 x - 1)^{\frac{n-1}{2}}$ & substitute $u = \sec x$

Other cases require creativity! & perhaps:

$$\int \tan x\,dx \overset{u=\cos x}{=} \ln|\sec x| + C$$

$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C = \ln|\cos x|^{-1} + C = \ln|\sec x| + C.$

$u=\cos x$, $\frac{du}{dx} = -\sin x$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

See textbook or check by differentiating

<u>Example</u> Find $\int \sin 6x \cos 5x\,dx$.

<u>Solution</u> In this case use (1) $\sin A \cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$

So $= \frac{1}{2}\int \sin x + \sin 11x\,dx$

$$= -\frac{1}{2}\left(\cos x + \frac{1}{11}\cos 11x\right) + C.$$

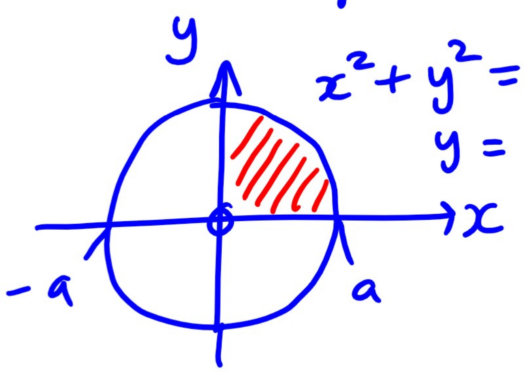
Also remember: (2) $\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$

(3) $\cos A \cos B = \frac{1}{2}\left(\cos(A-B) + \cos(A+B)\right)$

7.3 Trigonometric Substitution

Example: Use integration to find the area of a circle of radius a.

Solution



$x^2 + y^2 = a^2$

$y = \sqrt{a^2 - x^2}$

Area of $\frac{1}{4}$-circle

$$= \int_0^a \sqrt{a^2 - x^2}\, dx.$$

??? No obvious substitution

Trick: substitute $u = \sin^{-1}\left(\frac{x}{a}\right) \longrightarrow \sin u = \frac{x}{a}$

$\rightarrow a \sin u = x$

$$\frac{du}{dx} = \frac{1}{a} \cdot \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} = \frac{1}{a} \cdot \frac{1}{\sqrt{1 - \sin^2 u}}$$

Chain Rule

$$= \frac{1}{a \cos u}$$

$$dx = a \cos u\, du$$

$$\text{Area} = \int_{0 = \sin^{-1}\left(\frac{0}{a}\right)}^{\frac{\pi}{2} = \sin^{-1}\left(\frac{a}{a}\right)} \sqrt{a^2 - \underbrace{a^2 \sin^2 u}_{x^2}}\; a \cos u\, du = \int_0^{\pi/2} \sqrt{a^2 \underbrace{(1 - \sin^2 u)}_{\cos^2 u}}\; a \cos u\, du$$

Filter endpoints through substitution

$$= \int_0^{\pi/2} (a \cos u)(a \cos u)\, du$$

$$= \int_0^{\pi/2} a^2 \cos^2 u \, du$$

$\frac{1}{2}$ angle formula

$$= \int_0^{\pi/2} \frac{a^2}{2}(1+\cos \underbrace{2u}_{}) \, du$$

$v = 2u$, $dv = 2du$
$\int \cos 2u \, du = \frac{1}{2}\int \cos v \, dv$
$= \frac{1}{2}\sin v + C$
$= \frac{1}{2}\sin 2u + C$

$$= \frac{a^2}{2}\left[u + \frac{1}{2}\sin 2u\right]_0^{\pi/2}$$

$$= \frac{a^2}{2}\left(\frac{\pi}{2} + 0 - 0\right) = \frac{a^2\pi}{4}$$

So total area $= 4 \times \frac{\pi a^2}{4} = \pi a^2$.

No surprise there!!!

(But where did the formula come from originally?)