1A03 – CALCULUS I FOR SCIENCE
(SECTION C02)

Lecture 33

Last time — TRIGONOMETRIC SUBSTITUTION RULE:

① To simplify $\sqrt{a^2 - x^2}$ in an integral expression,*

substitute $\theta = \sin^{-1}\left(\frac{x}{a}\right)$ or better said, do an

ONLY VALID for $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

inverse substitution: $x = a\sin\theta$ $(dx = a\cos\theta)$

& use $1 - \sin^2\theta = \cos^2\theta$: $\sqrt{a^2 - x^2} = a\cos\theta$

* if other methods don't help

② To simplify $\sqrt{a^2 + x^2}$, let $\theta = \tan^{-1}\left(\frac{x}{a}\right)$ or better

$x = a\tan\theta$ valid for $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$dx = a\sec^2\theta \, d\theta$

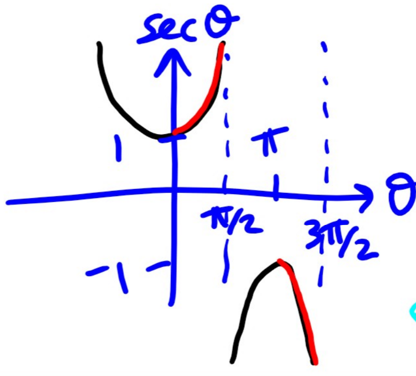
use $1 + \tan^2\theta = \sec^2\theta$

③ To simplify $\sqrt{x^2 - a^2}$, let $\theta = \sec^{-1}\left(\frac{x}{a}\right)$ or

(way) better $x = a\sec\theta$

use $\sec^2\theta - 1 = \tan^2\theta$

$$\frac{dx}{d\theta} = a\tan\theta\sec\theta$$



valid for θ in $\left[0, \frac{\pi}{2}\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$

we have to make a choice — this is the choice the textbook makes.

WARNING : BE VERY CAREFUL WITH ENDPOINTS

Last time $$\int_0^a \sqrt{a^2-x^2}\,dx = \int_{0=\sin^{-1}\left(\frac{0}{a}\right)}^{\pi/2=\sin^{-1}\left(\frac{a}{a}\right)} a^2\cos^2\theta\,d\theta$$

Remember to use the actual substitution here, not the inverse sub.

Example Find $$\int \frac{dx}{\sqrt{x^2+4x-5}}$$

Solution

Not in the right form so complete the square:

$$x^2+4x-5 = (x+2)^2-9$$

$$\int \frac{dx}{\sqrt{x^2+4x-5}} = \int \frac{dx}{\sqrt{(x+2)^2-9}} = \int \frac{du}{\sqrt{u^2-3^2}}$$

$u = x+2$
$du = dx$

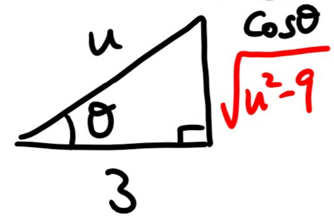
Inverse subs.:
$u = 3\sec\theta$
$\frac{du}{d\theta} = 3\sec\theta\tan\theta$

$$= \int \frac{3\sec\theta\tan\theta\,d\theta}{\sqrt{3^2\sec^2\theta-3^2}} = \int \frac{\cancel{3}\sec\theta\tan\theta}{\cancel{3}\sqrt{\sec^2\theta-1}}\,d\theta$$

$$= \int \frac{\sec\theta\,\cancel{\tan\theta}}{\cancel{\tan\theta}}\,d\theta = \int \sec\theta\,d\theta$$

$u = 3\sec\theta$

$\Rightarrow \dfrac{u}{3} = \sec\theta = \dfrac{1}{\cos\theta}$



$u = x+2$

$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \ln\left|\frac{u}{3} + \frac{\sqrt{u^2-9}}{3}\right| + C$$

$$= \ln\left|\frac{x+2}{3} + \frac{\sqrt{(x+2)^2-9}}{3}\right| + C$$

$$= \ln\left|\frac{x+2+\sqrt{x^2+4x-5}}{3}\right| + C$$

$$= \ln\left|x+2+\sqrt{x^2+4x-5}\right| \cancel{- \ln 3 + C} + C.$$

7.4 Integration of Rational Functions by Partial Fractions

$$f(x) = \frac{P(x)}{Q(x)} \quad \leftarrow \text{polynomials}$$

Want to integrate: $\displaystyle\int \frac{P(x)}{Q(x)}\,dx$

Step 1 If degree $(P) \geq$ degree(Q) ($\dfrac{P(x)}{Q(x)}$ "improper")

do long division to get:

$$\frac{P(x)}{Q(x)} = \underbrace{S(x)}_{\text{polynomial}} + \frac{R(x)}{Q(x)}$$

$\leftarrow$ remainder with degree (R) < degree (Q) ("proper")

<u>Example</u>

$$\frac{P(x)}{Q(x)} = \frac{2x^4+4x^3-11x^2-9x+16}{x^2+2x-3}$$

$$\begin{array}{r l}
 & 2x^2 - 5 \quad \leftarrow S(x) \\
\overset{Q(x)}{x^2+2x-3} \big) & 2x^4+4x^3-11x^2-9x+16 \quad \leftarrow P(x) \\
 & 2x^4+4x^3-6x^2 \\
\hline
 & \qquad\qquad -5x^2-9x+16 \\
 & \qquad\qquad -5x^2-10x+15 \\
\hline
 & \qquad\qquad\qquad\quad x+1 \quad \leftarrow R(x)
\end{array}$$

$$\frac{P(x)}{Q(x)} = \overset{S(x)}{2x^2-5} + \frac{x+1 \leftarrow R(x)}{x^2+2x-3 \leftarrow Q(x)}.$$

<u>Step 2</u> Factor $Q(x)$ into irreducible factors.

The "Fundamental Theorem of Algebra" tells us that all the factors will look like

$ax+b$ (linear)

or ax^2+bx+c (irreducible quadratic)
i.e. $b^2-4ac<0$.

<u>Example</u> $Q(x) = x^2+2x-3 = (x+3)(x-1)$.

<u>Step 3</u> 4 cases depending on factors of $Q(x)$

<u>Case I</u> $Q(x)$ has <u>distinct</u> linear factors <u>only</u>.

<u>Case II</u> $Q(x)$ has linear factors only, but at least one <u>repeats</u>.

<u>Case III</u> $Q(x)$ has at <u>least one irreducible quadratic factor</u> but no repeated irr. quad. factors

<u>Case IV</u> $Q(x)$ has at least one <u>repeated</u> irreducible quadratic factor.

<u>Case I</u> $Q(x)$ has distinct linear factors only.

From Step 2 we have $Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$

Then $\dfrac{R(x)}{Q(x)} = \dfrac{A_1}{a_1x+b_1} + \dfrac{A_2}{a_2x+b_2} + \ldots + \dfrac{A_k}{a_kx+b_k}$

for some #s A_i that we need to find.

called the "Partial Fraction Expression"

(This we will integrate.)

<u>Example</u> From above we got $\dfrac{R(x)}{Q(x)} = \dfrac{x+1}{(x-1)(x+3)}$

Set this equal to $\dfrac{A}{x+3} + \dfrac{B}{x-1}$

To find A, B:

Multiply by $Q(x)$: $\quad x+1 = A(x-1) + B(x+3)$

2 choices: ① multiply out & compare coefficients.

$x+1 = Ax - A + Bx + 3B = (A+B)x + (-A+3B) \Rightarrow 1 = A+B$, $1 = -A+3B$ $\Rightarrow B = \frac{1}{2}$, $A = \frac{1}{2}$.

② Plug in helpful x-values T.B.C.

(I wrote some more in class but the computer crashed in sympathy with you guys! So this step is repeated at the start of the next lecture.)