1A03 – CALCULUS I FOR SCIENCE (SECTION C02)

Lecture 34

Last time $\int \frac{P(x)}{Q(x)}\,dx$ by Partial Fractions

Step 1 Write $\frac{P(x)}{Q(x)} = S(x) + \boxed{\frac{R(x)}{Q(x)}}$ using e.g. long division

$S(x)$: polynomial; $R(x)$: remainder; degree(R) < degree(Q)

Step 2 Factorize $Q(x)$ into $ax+b$ and ax^2+bx+c factors ← $b^2-4ac<0$

Step 3 Case I If $Q(x)$ has only distinct, linear factors, write

PARTIAL FRACTION EXPRESSION → $\boxed{\frac{R(x)}{Q(x)}} = \frac{A_1}{a_1x+b_1} + \dots + \frac{A_k}{a_kx+b_k}$

← solve for $A_1, \dots, A_k$

← factors of $Q(x)$

Example continued We got $\frac{R(x)}{Q(x)} = \frac{x+1}{(x+3)(x-1)}$

Set this $= \frac{A}{x+3} + \frac{B}{x-1}$

Multiply by $Q(x)$:

$$x+1 = A(x-1) + B(x+3)$$

i.e. compare coefficients on both sides of the x-terms & the constant terms (if $mx+n = jx+k$, then $m=j$, $n=k$)

2 options: ① multiply out & compare coefficients & solve the equations you get for A & B

② equations holds for all x so "cover-up" terms with clever choices for x-values:

$x = 1 \longrightarrow 1+1 = B(1+3) \Rightarrow B = \frac{1}{2}$

$x = -3 \longrightarrow -3+1 = A(-3-1) \Rightarrow A = \frac{1}{2}$

Thus $\dfrac{R(x)}{Q(x)} = \dfrac{1}{2(x+3)} + \dfrac{1}{2(x-1)}$.

Now integrate:

$$\int \frac{P(x)}{Q(x)}\,dx = \int \frac{2x^4+4x^3-11x^2-9x+16}{x^2+2x-3}\,dx$$

$$= \int 2x^2 - 5 + \frac{R(x)}{Q(x)}\,dx$$

$$= \int 2x^2 - 5 + \frac{1}{2(x+3)} + \frac{1}{2(x-1)}\,dx$$

$$= \frac{2}{3}x^3 - 5x + \frac{1}{2}\ln|x+3| + \frac{1}{2}\ln|x-1| + C$$

<u>Case II</u> $Q(x)$ has linear factors only, but at least one repeats.

If $(ax+b)$ repeats r times, replace $\dfrac{A}{ax+b}$

with $\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \dots + \dfrac{A_r}{(ax+b)^r}$ in the

partial fraction expression for $\dfrac{R(x)}{Q(x)}$, then integrate.

<u>Example</u> Find $\displaystyle\int \frac{3x-2}{x^2(x-1)^2(x+2)}\,dx$.

Solution No need for Steps 1 or 2 here. (Already proper & $Q(x)$ fully factored.)

Step 3
Case II Write $\dfrac{3x-2}{x^2(x-1)^2(x+2)} = \dfrac{A_1}{x} + \dfrac{A_2}{x^2} + \dfrac{B_1}{(x-1)} + \dfrac{B_2}{(x-1)^2} + \dfrac{C}{x+2}$

$$\Rightarrow 3x-2 = A_1 x(x-1)^2(x+2) + A_2(x-1)^2(x+2) + B_1 x^2(x-1)(x+2) + B_2 x^2(x+2) + Cx^2(x-1)^2$$

2 options: ① multiply out & compare coefficients (x^4, x^3, x^2, x coefficients & constants)

② Choose clever x-values to "cover-up":

$x = 0 \dashrightarrow A_2 = -1$

$x = 1 \dashrightarrow B_2 = \frac{1}{3}$

$x = -2 \dashrightarrow C = -\frac{2}{9}$

To find A_1 and B_1, need 2 more equations.
Can go back to comparing x^0 coefficients <u>or</u> plug in 2 more x-values:

plug in values for A_2, B_2 and C

Say $x = -1$: $-5 = -4A_1 - 4 - 2B_1 + \frac{1}{3} - \frac{8}{9}$

$\Rightarrow 2A_1 + B_1 = 2/9$

$x = 2$: $A_1 + 2B_1 = 4/9$ (Exercise!)

$\Rightarrow A_1 = 0, \; B_1 = \frac{2}{9}.$

All together:

$$\int \frac{3x-2}{x^2(x-1)^2(x+2)}\,dx = \int \cancel{\frac{0}{x}} - \frac{1}{x^2} + \frac{2}{9(x-1)} + \frac{1}{3\underbrace{(x-1)}_{u=x-1}^2} - \frac{2}{9(x+2)}\,dx$$

$$= \frac{1}{x} + \frac{2}{9}\ln|x-1| - \frac{1}{3(x-1)} - \frac{2}{9}\ln|x+2| + C$$

<u>Case III</u> $Q(x)$ contains at least one irred. quadratic factor but no irred. quad. factor repeats.

For each ax^2+bx+c factor include

$$\frac{Ax+B}{ax^2+bx+c}$$ in partial fraction expr. for $\frac{R(x)}{Q(x)}$.

For linear terms $\rightarrow$ as in Cases I or II. Then integrate.

<u>Example</u> Find $\displaystyle\int \frac{x^2-3x+3}{(x+1)^2(2x^2-3x+2)}\,dx$.

$b^2-4ac = 9-4\cdot4 = -7 < 0$

<u>Solution</u> Steps 1 & 2 already done.

<u>Step 3</u> <u>Case III</u>. Write $\dfrac{x^2-3x+3}{(x+1)^2(2x^2-3x+2)} =$

$$\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{2x^2-3x+2}$$

Now: multiply through by $Q(x)$:

$$x^2 - 3x + 3 = A(x+1)(2x^2-3x+2) + B(2x^2-3x+2) + (Cx+D)(x+1)^2$$

2 options, but clever x-values won't get you so far.

So compare coeffs:

$$\left.\begin{aligned} x^3 &: \quad 0 = 2A + C \\ x^2 &: \quad 1 = -A + 2B + 2C + D \\ x &: \quad -3 = -A - 3B + C + 2D \\ \text{const} &: \quad 3 = 2A + 2B + D \end{aligned}\right\}$$

Solve this system: $A = \frac{2}{7}$, $B = 1$, $C = -\frac{4}{7}$, $D = \frac{3}{7}$.

So $$\int \frac{x^2-3x+3}{(x+1)^2(2x^2-3x+2)}\,dx = \int \frac{2}{7(x+1)}\,dx + \int \frac{dx}{(x+1)^2} + \int \frac{3-4x}{7(2x^2-3x+2)}\,dx$$

$u = 2x^2 - 3x + 2$

$du = (4x-3)\,dx$

$= \frac{2}{7}\ln|x+1| - \frac{1}{(x+1)}$

$$-\frac{1}{7}\ln|2x^2-3x+2|+C$$

$$\int \frac{-du}{7u}$$

$$= -\frac{1}{7}\ln|u| + C$$

$$= -\frac{1}{7}\ln|2x^2-3x+2| + C$$

This was luck that we could perform sub. directly - next time, in the guise of Case IV, we'll see what to do if things are not so nice here.