1A03 – CALCULUS I FOR SCIENCE
(SECTION C02)

Lecture 35

Last time $\int \frac{P(x)}{Q(x)}\,dx$ using PARTIAL FRACTIONS (ctd)

From $\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$ } we form a Partial Fraction Expression for $\frac{R(x)}{Q(x)}$ depending on factors of $Q(x)$.

e.g. Case III

$(ax+b)^r \to \frac{A_1}{ax+b} + \dots + \frac{A_r}{(ax+b)^r}$ and $ax^2+bx+c \to \frac{Ax+B}{ax^2+bx+c}$.

Case IV $Q(x)$ has at least one repeated irreducible quadratic factor.

If $(ax^2+bx+c)^r$ is a factor, then include

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \dots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

in the Partial Fraction Expression for $\frac{R(x)}{Q(x)}$, & integrate.

Example Find $\int \frac{3x^4-16x^3+42x^2-54x+23}{(x-3)(x^2-2x+2)^2}\,dx$

Solution Steps 1 & 2 already done.

Step 3 Case IV

$$\frac{A}{x-3} + \frac{Bx+C}{x^2-2x+2} + \frac{Dx+E}{(x^2-2x+2)^2}$$

Use computer package!

$$= \frac{2}{x-3} + \frac{x-3}{x^2-2x+2} + \frac{3x+1}{(x^2-2x+2)^2}$$

Now $\displaystyle\int \frac{2}{x-3}\,dx = 2\ln|x-3| + C$ ✓

As for $\displaystyle\int \frac{x-3}{\underbrace{x^2-2x+2}}\,dx$

↳ Want to substitute $u = x^2-2x+2$

$du = (2x-2)\,du$

↳ So we write

What we want

Some-thing simpler

$$\frac{x-3}{x^2-2x+2} = \boxed{\underset{K}{\text{Const.}}\,\frac{(2x-2)}{x^2-2x+2}} + \boxed{\frac{\overset{L}{\text{const.}}}{x^2-2x+2}}$$

↳ To find the constants write $x-3 = K(2x-2) + L$

& solve for K, L : $K = 1/2,\ L = -2$

So

$$\int \frac{x-3}{x^2-2x+2}\,dx = \frac{1}{2}\int \frac{2x-2}{\underbrace{x^2-2x+2}_{\text{Sub. } u = x^2-2x+2}}\,dx - 2\int \frac{dx}{x^2-2x+2}$$

$$= \frac{1}{2}\int \frac{du}{u}$$

$$= \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|x^2-2x+2| + C$$

$$= -2\int \frac{dx}{\underbrace{(x-1)}_{}^2+1}$$

sub : $u = x-1$
$du = dx$

$$= -2\int \frac{du}{u^2+1}$$

$$= -2\arctan(u) + C$$

$$= -2\arctan(x-1) + C.$$

Finally: $\displaystyle\int \frac{3x+1}{\underbrace{(x^2-2x+2)}_{}^2}\,dx$ — same ideas apply:

Again want to sub. $u = x^2-2x+2$
$du = 2x-2$

So again split up numerator: $3x+1 = M(2x-2)+N$

$\longrightarrow M = 3/2,\ N = 4$

What we want

Something simpler

$$\text{So } \int \frac{3x+1}{(x^2-2x+2)^2}\,dx = \frac{3}{2}\int \frac{2x-2}{\underbrace{(x^2-2x+2)}_{}^2}\,dx + 4\int \frac{dx}{(x^2-2x+2)^2}$$

Sub. $u = x^2-2x+2$
$du = 2x-2$

$$= \frac{3}{2}\int \frac{du}{u^2} = -\frac{3}{2u} + C$$

$$= \frac{-3}{2(x^2-2x+2)} + C$$

Complete the square

$$= 4\int \frac{dx}{(\underbrace{(x-1)}_{}^2+1)^2}$$

Sub. $u = x-1$
$du = dx$

$$= 4\int \frac{du}{(u^2+1)^2}$$

$u = \tan\theta$

$du = \sec^2\theta\, d\theta$

$$= 4\int \frac{\sec^2\theta}{(\tan^2\theta + 1)^2}\, d\theta = 4\int \frac{\sec^2\theta}{(\sec^2\theta)^2}\, d\theta$$

$$= 4\int \frac{1}{\sec^2\theta}\, d\theta = 4\int \cos^2\theta\, d\theta$$

$$= \frac{4}{2}\int \cos 2\theta + 1\ d\theta = \sin 2\theta + 2\theta + C$$

$$= \underbrace{\frac{2u}{u^2+1}} + 2\arctan(u) + C$$

Exercise

If $u = \tan\theta$ what is $\sin 2\theta$?

$$= \frac{2(x-1)}{x^2 - 2x + 2} + 2\arctan(x-1) + C.$$

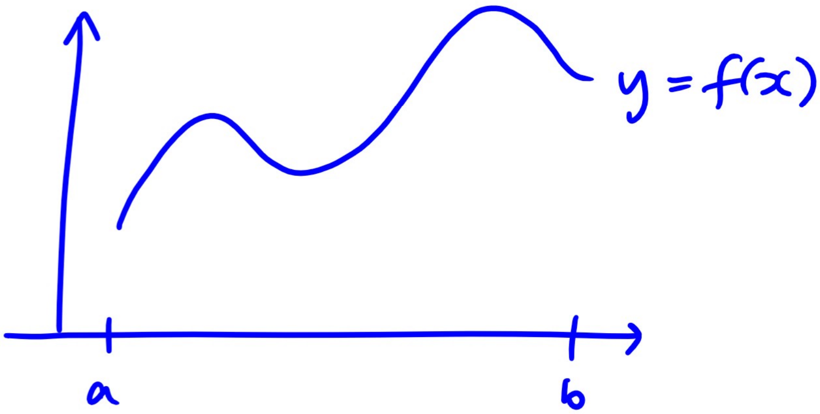
FINAL ANSWER $= 2\ln|x-3| + \frac{1}{2}\ln|x^2-2x+2|$

$$- \cancel{2\arctan(x-1)} + \frac{4x-7}{2(x^2-2x+2)}$$

$$+ \cancel{2\arctan(x-1)} + C$$

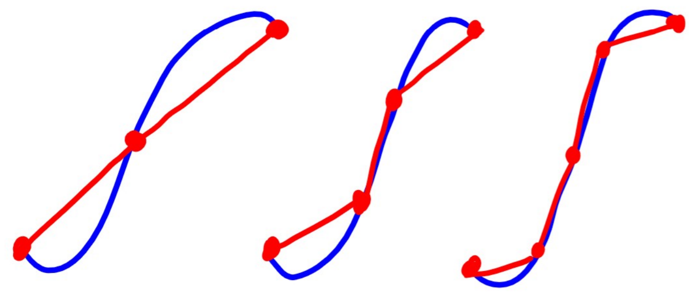
One final application of Integration!

8.1 Arc Length

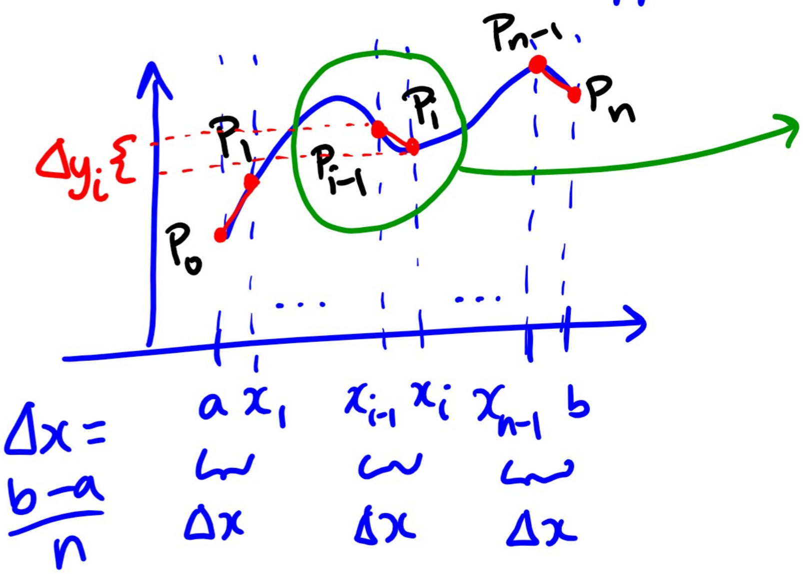


How do we find the "arc length" of the curve $y = f(x)$?

Can approx. by line segments:



The more line segments, the better the approximation.



What is $|P_{i-1}P_i|$, the length of the segment from P_{i-1} to P_i?

$$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

depends on i:

$$\Delta y_i = f(x_i) - f(x_{i-1})$$

By M.V.T. there is <u>some</u> point $c_i \in [x_{i-1}, x_i]$ with

$$\left\{ f'(c_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \right\}$$ rearrange this

So $\Delta y_i = f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1})$

$= f'(c_i)\Delta x$

So $|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + f'(c_i)^2(\Delta x)^2} = \left[\sqrt{1+f'(c_i)^2}\right]\Delta x.$

T. B. C.