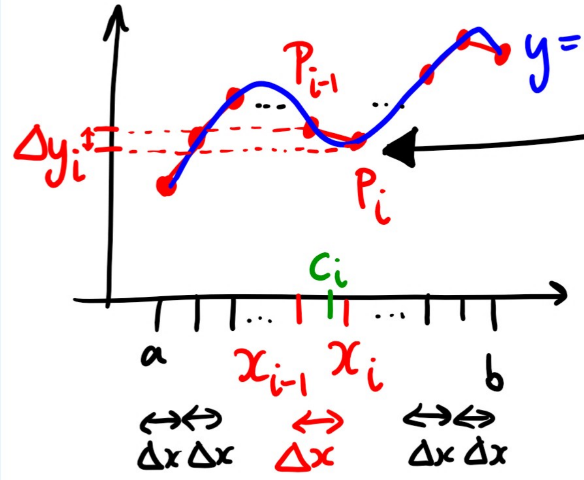
1A03 – CALCULUS I FOR SCIENCE

(SECTION C02)

Lecture 36

Last time: Arc Length – What is the length of $y=f(x)$?

So far:



$|P_{i-1}P_i|$ = length of line segment $P_{i-1} \to P_i$

$$= \sqrt{1+(f'(c_i))^2} \cdot \Delta x$$

[where $f'(c_i) = \frac{f(x_i)-f(x_{i-1})}{x_i - x_{i-1}} = \frac{\Delta y_i}{\Delta x}$]
(M.V.T.)

So length of $y=f(x)$ from $x=a$ to $x=b$ $\approx \sum_{i=1}^{n} |P_{i-1}P_i|$

The sum of all the line segments approximates the length of the curve.

$$= \sum_{i=1}^{n} \sqrt{1+(f'(c_i))^2}\,\Delta x$$

Thus length of $y=f(x)$ from $x=a$ to $x=b$ $= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+f'(c_i)^2}\,\Delta x$ if limit exists, say $f'(x)$ continuous

We think of c_i as a sample point & x_{i-1}, c_i & x_i all end up in the same place squished together.

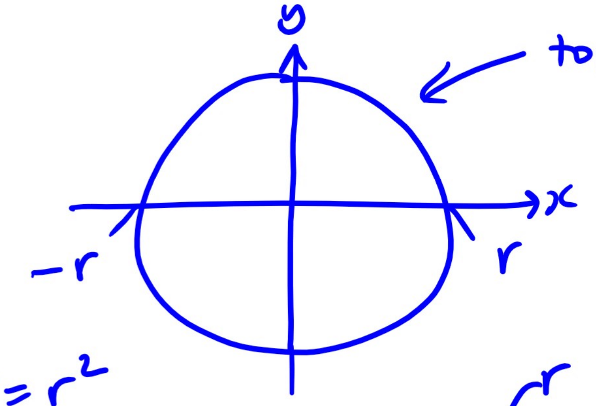
$$= \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

if $f'(x)$ is continuous.

$$\left(= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \quad \text{if you prefer.} \right)$$

Example Show why the circumference of a circle of radius r is $2\pi r$.

Solution



top $\frac{1}{2}$: $y = \sqrt{r^2 - x^2} = (r^2 - x^2)^{1/2}$

$x^2 + y^2 = r^2$

So circumference of top $\frac{1}{2}$ of circle =

$$\int_{-r}^{r} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = \frac{1}{2}(-2x)(r^2 - x^2)^{-1/2}$$

$$= \frac{-x}{\sqrt{r^2 - x^2}}$$

$$= \int_{-r}^{r} \sqrt{1 + \left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\, dx$$

$$= \int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx$$

$$= \int_{-r}^{r} \sqrt{\frac{r^2 \cancel{- x^2} \cancel{+ x^2}}{r^2 - x^2}}\, dx = \int_{-r}^{r} \frac{r}{\sqrt{r^2 - x^2}}\, dx$$

or say $r = r\sin\theta \rightarrow$ what is θ? ← for top

$-r = r\sin\theta \rightarrow$ what is θ? ← for bottom

$$= r \int_{-r}^{r} \frac{dx}{\sqrt{r^2 - x^2}} = r \int_{\arcsin(-\frac{r}{r})}^{\arcsin(\frac{r}{r})} \frac{r\cos\theta\, d\theta}{\sqrt{r^2 - r^2\sin^2\theta}}$$

trig. sub.

$x = r\sin\theta$, $dx = r\cos\theta\, d\theta$

$$= r\int_{-\pi/2}^{\pi/2} \frac{r\cos\theta}{\sqrt{r^2\underbrace{\cancel{(1-\sin^2\theta)}}_{\cos^2\theta}}}\,d\theta = r\int_{-\pi/2}^{\pi/2} \frac{\cancel{r\cos\theta}}{r\,\cancel{\cos\theta}}\,d\theta = r\int_{-\pi/2}^{\pi/2} d\theta$$

$$= r\Big[\theta\Big]_{-\pi/2}^{\pi/2} = r\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

$$= \pi r$$

So total circumference of circle = 2 × top $\frac{1}{2}$ circum. = $\underline{\underline{2\pi r.}}$

Just as there's an "area up to x function" $\hookrightarrow \int_a^x f(t)\,dt$

we have an "arc length up to x function"

AKA "<u>arc length function</u>"

$$s(x) = \int_a^x \sqrt{1+(f'(t))^2}\,dt$$

, the length of $y=f(t)$ from $(a, f(a))$ to $(x, f(x))$

So, by F.T.C. Part I, $s'(x) = \sqrt{1+(f'(x))^2}$.

<u>Example</u> Find the arc length function for $y = \ln(\cos x)$ for $0 \le x \le \pi/4$.

$y = \ln(\cos(t))$

$$\frac{dy}{dt} = \frac{-\sin(t)}{\cos(t)}$$

$$= -\tan(t)$$

<u>Solution</u>

$$s(x) = \int_0^x \sqrt{1+((\ln(\cos(t)))')^2}\,dt$$

$$= \int_0^x \sqrt{1+(-\tan(t))^2}\,dt$$

$$= \int_0^x \sec(t)\,dt = \Big[\ln|\sec(t)+\tan(t)|\Big]_0^x$$

$$= \ln|\sec(x)+\tan(x)|$$

$$- \underbrace{\ln|\underset{1}{\cancel{\sec}(0)} + \underset{0}{\cancel{\tan}(0)}|}_{=0}$$

$$= \ln|\sec(x)+\tan(x)|.$$

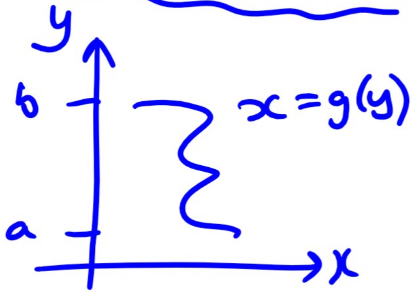
↑

NO + C – definite integral.

* * *

Notice : - take a curve $x = g(y)$

- roles switched so just get :



arc length from $y=a$ to $y=b$ $= \displaystyle\int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

<u>Example</u> Set up the integral for computing the arc length of $x = y - \sqrt{y^3}$ from $y=1$ to 2.

<u>Solution</u> $x = y - y^{3/2}$ so $\dfrac{dx}{dy} = 1 - \frac{3}{2}y^{1/2}$

Hence arc length $= \displaystyle\int_1^2 \sqrt{1+\left(1-\tfrac{3}{2}y^{1/2}\right)^2}\,dy = \int_1^2 \sqrt{1+1-3\sqrt{y}+\tfrac{9}{4}y}\,dy$

$$= \int_1^2 \sqrt{2 - 3\sqrt{y} + \frac{9}{4}y} \ dy .$$

↳ Hard (that's typical)

(for arc length integrals).

7.5 Integration Strategy (Read 7.5!)

First, <u>know</u> your basic antiderivatives (see table p. 503 of textbook)

There are new basic ones hiding there.

e.g. $\int \operatorname{cosec}^2(x)\, dx = -\cot(x) + C$

Next here are some useful guidelines to orient you:

① <u>Simplify</u>! Multiply out brackets, rewrite trig. functions (e.g. $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\sec\theta = \frac{1}{\cos\theta}$), Cancel.

② <u>Substitution</u> Look for obvious ones: the $u = g(x)$ where $g'(x)$ is a factor in the integrand e.g.

$$(u = x^2+5) \qquad \int x\,(x^2+5)^{3/2}\, dx$$

③ Classify (a) Trig. Integrals (7.2)

(b) Rational Functions (7.4)

(c) Radicals → $\sqrt{\pm a^2 \pm x^2}$ → trig. sub. (7.3)

↓

$\sqrt[n]{x}$ → try sub. $u = \sqrt[n]{x}$

(d) Integration by Parts i.e. $u\,dv$

gets simpler when we differentiate e.g. x^2, x^3, $\ln(x)$, $\arcsin(x)$, $\arctan(x)$ ↗ u

dv ↖ can integrate e.g. $\sin(x)$, $\cos(x)$, or e^x, or anything in the table

④ If at first you don't succeed, try again — but be smarter.

T.B.C.