1A03 – Calculus I for Science
(Section C02)

Lecture 6

Last time

Continuity: A function $f(x)$ is continuous at $x=a$ if $\underbrace{\lim_{x\to a} f(x) = f(a)}$.

This says that $\lim_{x\to a} f(x)$ exists ($\leftarrow$ is a #)

& that a is in the domain of $f(x)$ (so "$f(a)$" makes sense)

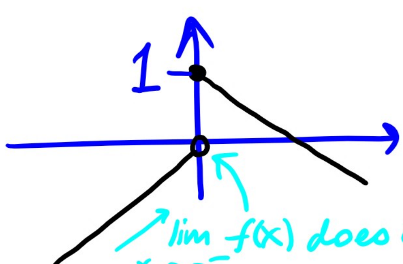
& that the two #s ($f(a)$ and the limit) are equal.

Discontinuities

- jump disconuities
- infinite "
- removable "

Jump disconuity:

$$f(x) = \begin{cases} x & x<0 \\ 1-x & x\geq 0 \end{cases}$$



$\lim_{x\to 0^-} f(x)$ does exist, but is 0, which is not the value of $f(0)$.

Note!

NB $f(x)$ continuous at $x=a$ for $a\neq 0$

At $x=0$, $\lim_{x\to 0} f(x)$ DNE

but notice

$$\lim_{x\to 0^-} f(x) = 0$$

$$\lim_{x\to 0^+} f(x) = 1 = f(0)$$

(i.e. both exist)

We say $f(x)$ is right continuous at $x=0$.

In general:

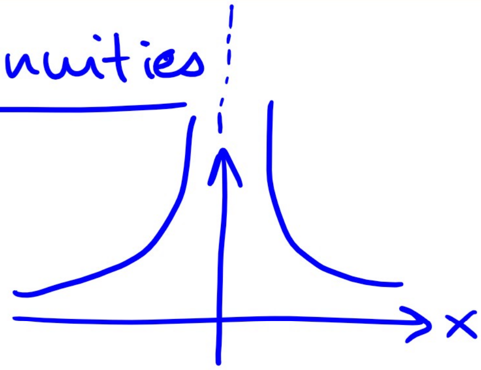
$\lim_{x\to a^-} f(x) = f(a) \longrightarrow f(x)$ is left continuous at $x=a$

$\lim_{x\to a^+} f(x) = f(a) \longrightarrow f(x)$ is right continuous.

(Both left & right continuous $\Rightarrow$ continuous.)

Infinite discontinuities

$$f(x) = \frac{1}{x^2}$$



$$\lim_{x \to 0^-} f(x) = \frac{1}{0^+} = \frac{1}{\text{small} +\text{ve}} = \text{big} +\text{ve} = \infty \quad \underline{\text{DNE}}$$

Also $f(x)$ not defined at $x = 0$.

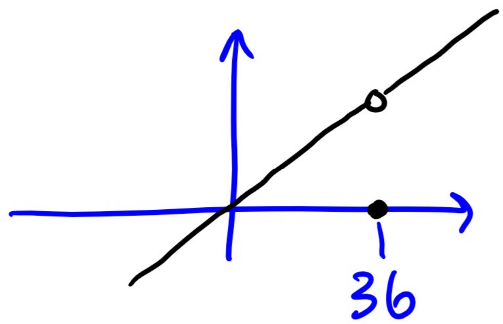
$$\& \lim_{x \to 0^+} f(x) = \infty \quad \underline{\text{DNE}}$$

So $\lim_{x \to 0} f(x)$ DNE.

So (definitely) not continuous at $x = 0$.

Removable Discontinuity

$$f(x) = \begin{cases} x & x \neq 36 \\ 0 & x = 36 \end{cases}$$



$$\lim_{x \to 36^-} f(x) = 36 = \lim_{x \to 36^+} f(x)$$

So $\lim_{x \to 36} f(x) = 36$ (i.e. exists!)

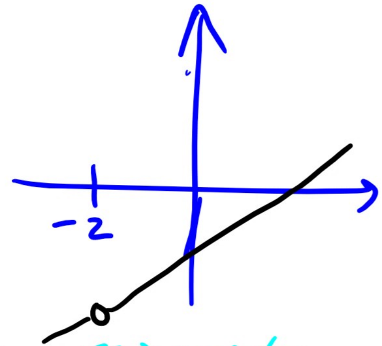
BUT $\neq f(36) = 0$

So notice: $f(x)$ NOT continuous at $x = 36$; also NOT right cont. & NOT left continuous.

Now look at $\cancel{f(x) = \frac{x^2+6}{x+8}}$ $f(x) = \dfrac{x^2 - x - 6}{x+2} = \dfrac{(x+2)(x-3)}{x+2}$

Everywhere except $x = -2$, $f(x) = x - 3$.

But $f(x)$ undefined at $x = -2$



Here $\lim_{x \to -2^-} f(x)$ exists and equals $\lim_{x \to -2^+} f(x)$ ← So $\lim_{x \to -2} f(x)$ exists.

BUT $f(x)$ undefined at $x = -2$ so $f(x)$ NOT continuous (or left or right continuous) at $x = -2$

(these limits would have to equal "$f(-2)$" but f not defined at $x = -2$)

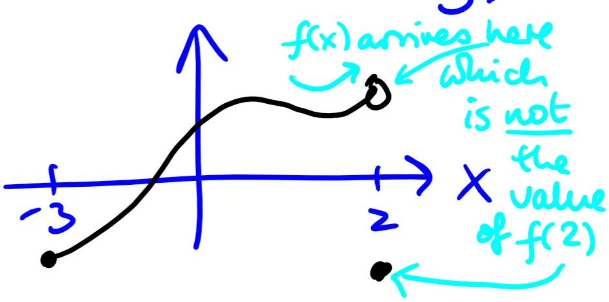
A function $f(x)$ is continuous on an interval (finite or infinite) if it is continuous at every point a in the interval.

↑ !!!

$[c, d)$

If the point a is an endpoint of the interval e.g. c then we ask for one-sided continuity (within the interval).

e.g.



f is cont. everywhere between -3 and 2

f is cont. from right at -3

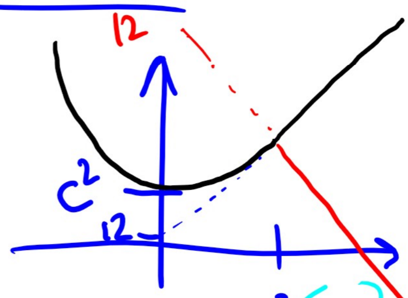
f is not cont. from left at 2

So we say f is continuous on interval $[-3, 2)$.

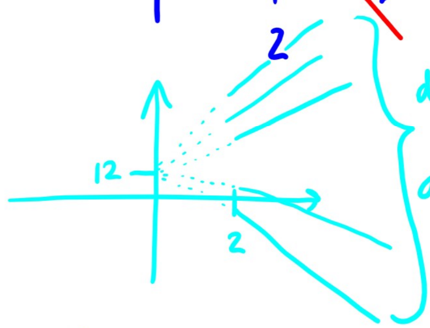
Functions that are continuous everywhere they are defined:

- polynomials
- rational functions
- root functions e.g. $\sqrt[5]{x}$, defined for $x \geq 0$
- trig. functions
- inverse trig. functions
- exponential functions
- log. functions

Example Find c that makes $f(x) = \begin{cases} x^2 + c^2 & \text{for } x < 2 \\ cx + 12 & \text{for } x \geq 2 \end{cases}$



We need $x^2 + c^2 = cx + 12$ when $x = 2$



different possiblities as c varies for cx+12 on x≥2.

i.e. $4 + c^2 = 2c + 12$

$$c^2 - 2c - 8 = 0$$

$$(c+2)(c-4) = 0$$

So we can take $c = -2$ or $c = 4$.

If $f(x), g(x)$ are continuous at $x = a$ then

so is $f(x) + g(x)$, $f(x) - g(x)$, $f(x)g(x)$,

$\dfrac{f(x)}{g(x)}$ if $g(a) \neq 0$.

<u>Composition</u> If $f(x)$, $g(x)$ with $b = \lim_{x \to a} g(x)$

and f continuous at b

Then $\lim_{x \to a} f(g(x)) = f(b)$ $\left(\text{i.e.} = f\left(\lim_{x \to a} g(x) \right) \right)$

so you can swap limit & application of f.

In particular if $b = g(a)$ i.e. g cont. at a, then this says $\lim_{x \to a} f(g(x)) = f(g(a))$

i.e. "a cont. function of a cont. function is continuous."

Intermediate Value Theorem

"Continuous functions can't jump."