1A03 – CALCULUS I FOR SCIENCE
(SECTION C02) Lecture 7

Last time Continuity

"The composition of continuous functions is continuous"

i.e. if $g(x)$ is cont. at a and $f(x)$ is cont. at $g(a)$,
then $(f\circ g)(x) = f(g(x))$ is cont. at a.

Also, coming soon... Intermediate Value Theorem
"Continuous functions can't jump."

Exercise When is $\ln(1+\sin(x))$ continuous?

Solution $\ln$ cont. where defined i.e. for inputs >0

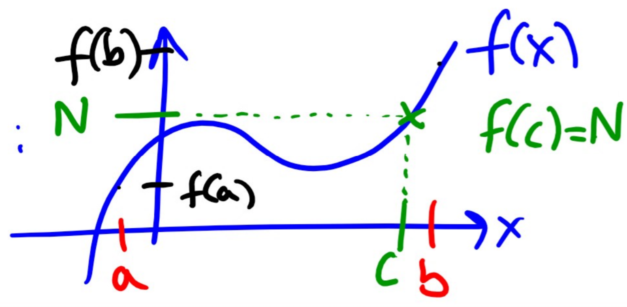
So where is $1+\sin(x) > 0$?
i.e. $\sin(x) > -1$?

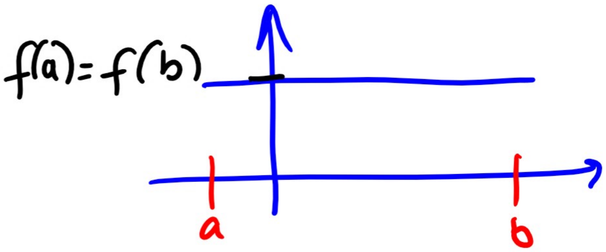
i.e. NOT when $\sin(x) = -1$ ($\sin(x)$ takes values in $[-1,1]$)

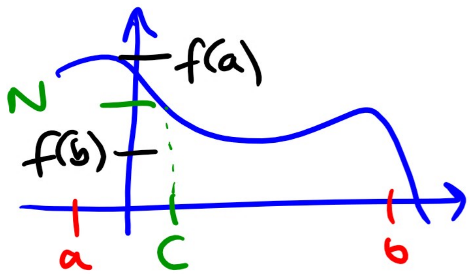
i.e. NOT at $x = \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots, \frac{-\pi}{2}, \frac{-5\pi}{2}, \ldots$

Intermediate Value Theorem

$f(x)$ continuous on $[a,b]$:







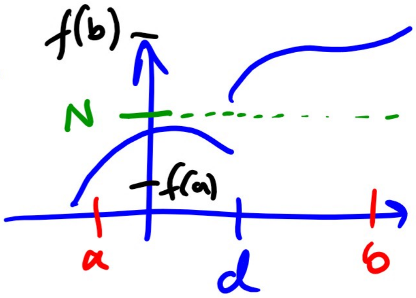
$f(c) = N$

Int. Value Theorem (IVT)

Think of it as a checklist: if you know ① and ②, then the conclusion is true too.

If ① $f(x)$ is continuous on $[a,b]$ and ② N is any intermediate value between $f(a)$ and $f(b)$ (inclusive), **then** there is some x-value c in $[a,b]$ with $f(c) = N$.

i.e. NOT:



$f(x)$ – discontinuous at d ← So ① is false

So ② is true → N is in between $f(a)$ and $f(b)$, but there is no x-value in $[a,b]$ with $f(x) = N$.

Example Use IVT to show that

$$f(x) = e^{\sin\left(\frac{\pi x}{2}\right)} - x$$

has a root/zero in $[1,2]$.

Solution To use IVT, need to know:

① – $f(x)$ cont. ✓ (see list above) – of functions cont. where defined

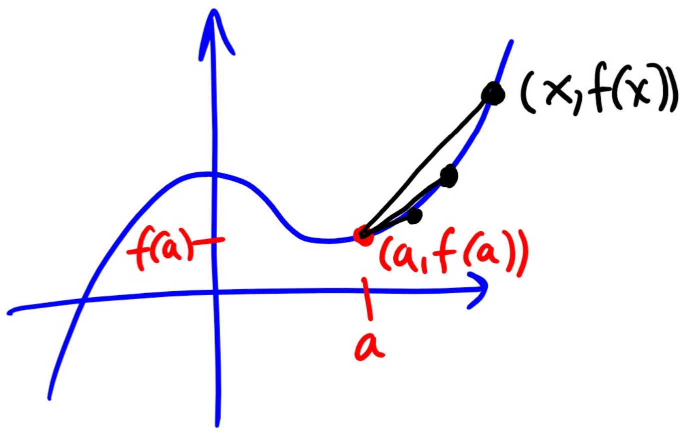
② – $N = 0$ is between $f(1)$ and $f(2)$.

$$f(1) = e^{\sin(\pi/2)} - 1 \simeq 2.7 - 1 = 1.7 > 0$$

$$f(2) = e^{\sin(\pi)} - 2 = 1 - 2 = -1 < 0$$

Since 0 lies between $f(1)$ and $f(2)$ and since $f(x)$ is cont. on $[1,2]$, by IVT there is a zero/root in $[1,2]$, i.e. some $c \in [1,2]$ with $f(c) = 0$.

2.7 Derivatives



The tangent line to $y = f(x)$ at the point $(a, f(a))$ is the line through $(a, f(a))$ with slope

$$m = \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right)$$

$$= \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right).$$

i.e. in both cases both one-sided limits exist

In Physics if $f(x)$ is position at time x, then m is velocity $v(a)$ at time $x = a$.

<u>Example</u> Find the equation of the tangent line to $f(x) = \frac{5}{x}$ at the point $(5, 1)$. ($5 = a$)

"<u>Solution</u>" Tangent line: $mx + c$.

Find $m = \lim_{h \to 0} \frac{f(5+h) - f(5)}{h} = \lim_{h \to 0} \frac{\left(\frac{5}{5+h} - 1\right)}{h}$

$$= \lim_{h \to 0} \left(\frac{5 - (5+h)}{h(5+h)} \right) \ldots\ldots = -\frac{1}{5}.$$

Fill in the gaps yourself!

Find c using $mx + c$ i.e. $-\frac{1}{5}x + c$ goes through $(5, 1)$.

↓ Fill in the gap yourself!

(Should get $y = -\frac{1}{5}x + 2$.)

The <u>derivative</u> of a function $f(x)$ at a # a is the slope of the tangent line at $(a, f(a))$, denoted $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

This is the instantaneous rate of change of $f(x)$ at $x = a$.

<u>Example</u> Find $f'(a)$ if $f(x) = 2x^2 - 3x + 7$.

<u>Solution</u>

$$f'(a) = \lim_{h \to 0} \left(\frac{2(a+h)^2 - 3(a+h) + 7 - (2a^2 - 3a + 7)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{\cancel{2a^2} + 4a\cancel{h} + 2h^{\cancel{2}} - \cancel{3a} - 3\cancel{h} \cancel{+7} \cancel{-2a^2} \cancel{+3a} \cancel{-7}}{\cancel{h}} \right)$$

$$= \lim_{h \to 0} (4a + 2h - 3) = 4a - 3.$$

<u>Example</u> Consider $\lim_{h \to 0} \frac{\sqrt[5]{32+h} - 2}{h}$?? For which $f(x)$ and point a is this limit $f'(a)$?

<u>Solution</u> We need $\sqrt[5]{32+h} - 2$ is $f(a+h) - f(a)$ for some $f(x)$ and some a.

Pair up terms if you can.

Can we find $f(x)$ & a with $\sqrt[5]{32+h} = f(a+h)$
$2 = f(a)$?

$f(x) = \sqrt[5]{x}$, $a = 32$.

This we see is the answer when we break the problem down into manageable pieces.