1A03 – Calculus I for Science
(Section C02)

Lecture 8

Last time: **Derivatives**

The derivative of a function $f(x)$ at $x=a$ is
$$f'(a) = \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\right).$$
(Slope of tangent line to $y=f(x)$ at $(a, f(a))$.)

Exercise Sketch the graph of $f(x)$ if

$f(0) = f(2) = f(3) = 0$ and these are the only roots

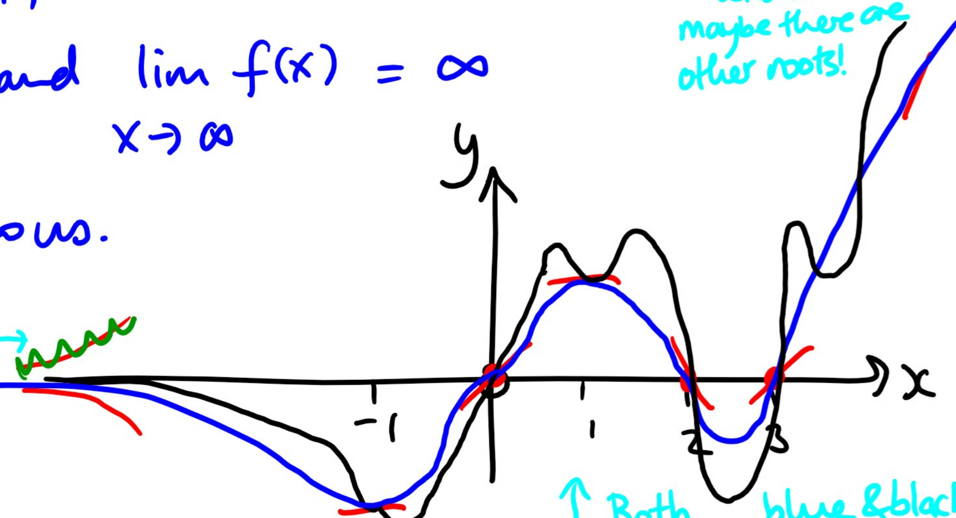
and $f'(0) = f'(3) = 1$ and $f'(2) = -1$

and $f'(1) = f'(-1) = 0$

and $\displaystyle\lim_{x\to-\infty} f(x) = 0$ and $\displaystyle\lim_{x\to\infty} f(x) = \infty$

and $f(x)$ is continuous.

I added an extra condition to turn this into an easier problem for us, but if this green text is not there, maybe there are other roots!



This is ruled out when we make the assumption written in green

Both blue & black curves are possibilities

2.8 The Derivative as a Function

Notice that $f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$

(So allow "a" to vary – now a variable "x")

is a function of x.

<u>Definition</u> We say that $f(x)$ is <u>differentiable</u> at $x=a$ if $f'(a)$ exists.

And we say that $f(x)$ is differentiable on an open interval (a,b) (where we could have $a=-\infty$ or $b=\infty$) if $f(x)$ is differentiable at every point in (a,b).

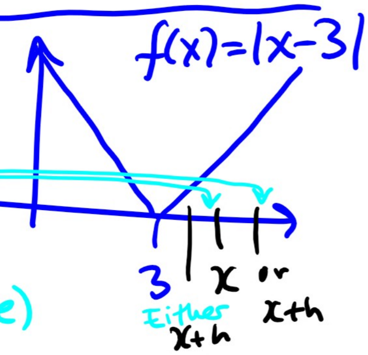
<u>3 ways functions can fail to be differentiable at a point</u>

<u>Example</u> Where is $f(x)=|x-3|$ differentiable?

Let's be clear: $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

(e.g. $|2|=2$, $|-3|=-(-3)=3$

<u>Solution</u>



For $x>3$, h small means $x+h>3$ as well (even if h negative)

So $\lim\limits_{h\to 0}\left(\dfrac{|x+h-3|-|x-3|}{h}\right)$

$= \lim\limits_{h\to 0}\left(\dfrac{\cancel{x}+h\cancel{-3}-\cancel{x}\cancel{+3}}{h}\right) = 1$ use this

$f(x)=\begin{cases} x-3, & x\geq 3 \\ -x+3, & x<3 \end{cases}$ by analogy with the above.

Similarly for $x < 3$, $\lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right) = -1$

What about $x = 3$?

If $h > 0$ ← so $|h| = h$, $\frac{|\overbrace{3+h-3}^{h}| - |\overbrace{3-3}^{0}|}{h} = \frac{|h|}{h} \overset{h \text{ (see above) or left}}{=} 1$

If $h < 0$, $\frac{f(3+h) - f(3)}{h} = -1$ ← $= \frac{|h|}{h} = \frac{-h}{h}$ (see above)

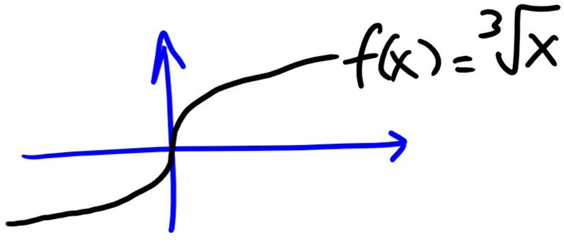
So $\lim_{h \to 3^-} \left(\frac{f(3+h) - f(3)}{h} \right) \neq \lim_{h \to 3^+} \left(\frac{f(3+h) - f(3)}{h} \right)$ So $f'(x)$ not defined at $x = 3$.

So $\lim_{h \to 3} \left(\frac{f(3+h) - f(3)}{h} \right)$ not defined.

① $f(x)$ NOT differentiable because of a "corner" / "kink" where left & right limits do not agree (as in previous example).

Example $f(x) = \sqrt[3]{x}$. Where is $f(x)$ differentiable?

Solution Draw graph: reflect graph of $y = x^3$ in line $y = x$



If we work out the limit, we get $f'(x) = \frac{1}{3(\sqrt[3]{x})^2}$ if $x \neq 0$

So $\lim_{x \to 0} f'(x) = \frac{1}{\text{small +ve}} = \infty$ DNE

② "Vertical tangent"

Notice $f(x) = |x-3|$ and $f(x) = \sqrt[3]{x}$ are both continuous, but still not diff^ble at (at least) one point. Fortunately we have:

This actually not needed for these two examples

<u>Theorem</u> If $f(x)$ is differentiable at $x=a$, then $f(x)$ is continuous at $x=a$.

(Justification: see textbook pp. 156-157)

↳ Not hard to understand, just uses a trick.

③ Theorem tells us that if $f(x)$ has a discontinuity at $x=a$, then $f(x)$ is NOT diferentiable at $x=a$. (If it were differentiable, it would have to be continuous!)

<u>Notation</u> Sometimes $f'(x)$ is written as

$$\frac{df}{dx} \quad \text{or} \quad \frac{d}{dx}f(x)$$

finding derivative of $f(x)$ = "<u>differentiating</u>" $f(x)$

So $\frac{d}{dx}$ is the "operation" of differentiating.

or if $y=f(x)$,

as $\frac{dy}{dx}$ or y'

<u>Higher Derivatives</u> We can keep taking derivatives ($f'(x)$ is

a function after all !)

[if defined]

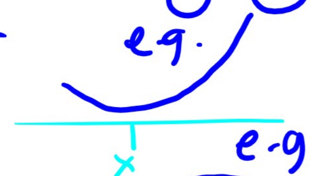
e.g. <u>2nd derivative</u>

$$f''(x) = \frac{d^2 f}{dx^2} = \frac{d^2}{dx^2} f(x) = y'' = \frac{d^2 y}{dx^2}$$

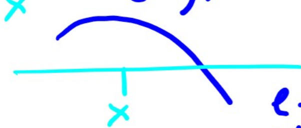
rate of change of $f'(x)$ at x

i.e. $f''(x)$ tells us how <u>slope</u> of $f(x)$ is changing

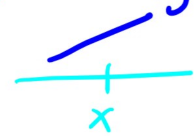
So $f''(x) > 0$ — getting steeper e.g.



$f''(x) < 0$ — getting less steep e.g.



$f''(x) = 0$ — slope stays same at x e.g.



We can keep going : $\frac{d^n f}{dx^n} = f^{(n)}(x)$ for "nth derivative".

We'll look at examples soon when we have some more concrete examples in mind (Chapter 3).