1A03 — Calculus I for Science
(Section C02)

Lecture 9

Last time Differentiability

A function $f(x)$ is differentiable at a if $f'(a)$ exists and $f(x)$ is differentiable on (a,b) if $f'(c)$ for every $c \in (a,b)$.

↳ Fails if $f(x)$ has a corner, or a vertical tangent, or if $f(x)$ is NOT continuous at $x=a$.

3.1, 3.2 — Building derivatives of complicated functions from simpler ones

Useful rules about differentiating:

If $f(x), g(x)$ differentiable at $x=a$, then so are $f(x) \pm g(x)$, $f(x)g(x)$, $cf(x)$ (for a constant c)

&

$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$ Sum/Difference Rules

$$(cf(x))' = cf'(x)$$ Constant Multiple Rule

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$ Product Rule (also Leibniz Rule)

Also $\dfrac{f(x)}{g(x)}$ is differentiable at $x=a$ (if f and g are) if $g(x) \neq 0$ and

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$ Quotient Rule

<u>Polynomials</u> The simplest polynomial : $y = c$ (constant!)

$y' = 0$ (zero slope!)

Next simplest : $y = x$. Here $y' = 1$.

Then : $y = x^2$. $y' = (x.x)' = x'.x + x.x'$
$= x + x = 2x$

Then : $y = x^3$. $y' = (x^2.x)' = (x^2)'.x + x^2.(x)'$
$= (2x).x + x^2$
$= 3x^2$

$\vdots$

$$\boxed{\text{If } f(x) = x^n, \text{ then } f'(x) = nx^{n-1}}$$

Power Rule.

→ In fact this works for any power n (not just positive integers).

Example Differentiate $f(x) = \dfrac{1}{\sqrt[5]{x^3}} - \dfrac{1}{\sqrt{x}}$

Solution

$$f'(x) = \left(\frac{1}{\sqrt[5]{x^3}}\right)' - \left(\frac{1}{\sqrt{x}}\right)'$$

$$= \left(\frac{1}{x^{3/5}}\right)' - \left(\frac{1}{x^{1/2}}\right)'$$

$$= \left(x^{-3/5}\right)' - \left(x^{-1/2}\right)'$$

$$= -\frac{3}{5}x^{-3/5-1} - \left(-\frac{1}{2}x^{-1/2-1}\right)$$

$$= -\frac{3}{5}x^{-8/5} + \frac{1}{2}x^{-3/2}.$$

General form of a polynomial:

$$f(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_d x^d$$

where d = degree of $f(x)$.

So $f'(x) = 0 + c_1 + 2c_2 x + \ldots + d c_d x^{d-1}$.

Example $3x^{100} + 2x^{36} - 5x^2 = f(x)$

Solution $f'(x) = 300x^{99} + 72x^{35} - 10x.$

Exponential Functions $f(x) = b^x$, $b>0$, $b \neq 1$

$$\frac{df}{dx}(x) = \lim_{h\to 0} \frac{b^{x+h} - b^x}{h}$$

$$= \lim_{h\to 0} \frac{b^x(b^h - 1)}{h} = b^x \underbrace{\lim_{h\to 0} \frac{b^h - 1}{h}}_{}$$

$= f'(0)$ IF this limit exists

You can assume it does!!

So $\frac{df}{dx}(x) = b^x \frac{df}{dx}(0)$.

By definition Euler's constant e satisfies

$$\left(\frac{d}{dx}(e^x)\right)(0) = 1 \quad \text{so}$$

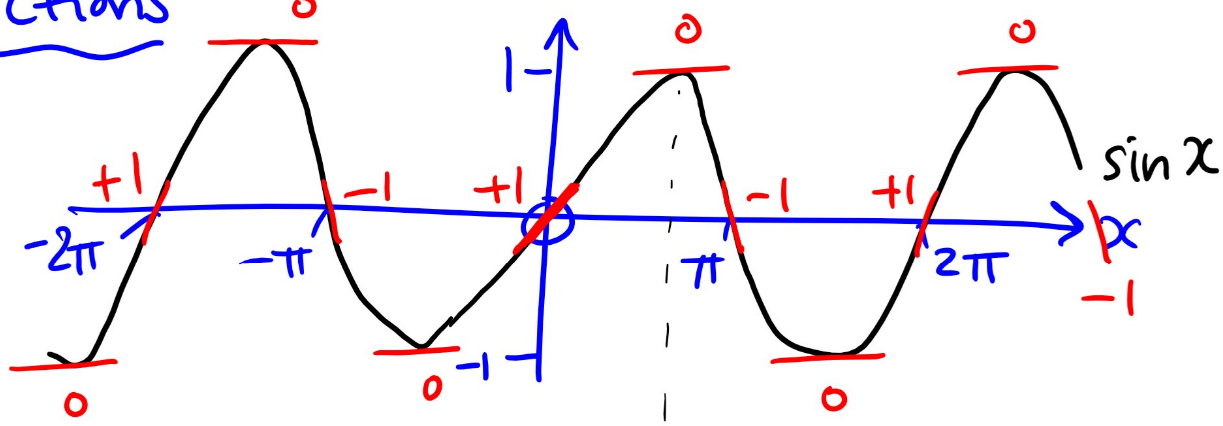
for $f(x) = e^x$, we have $\frac{df}{dx} = e^x$

Example Differentiate $f(x) = xe^x$.

Solution $f'(x) = (x)'e^x + x(e^x)'$

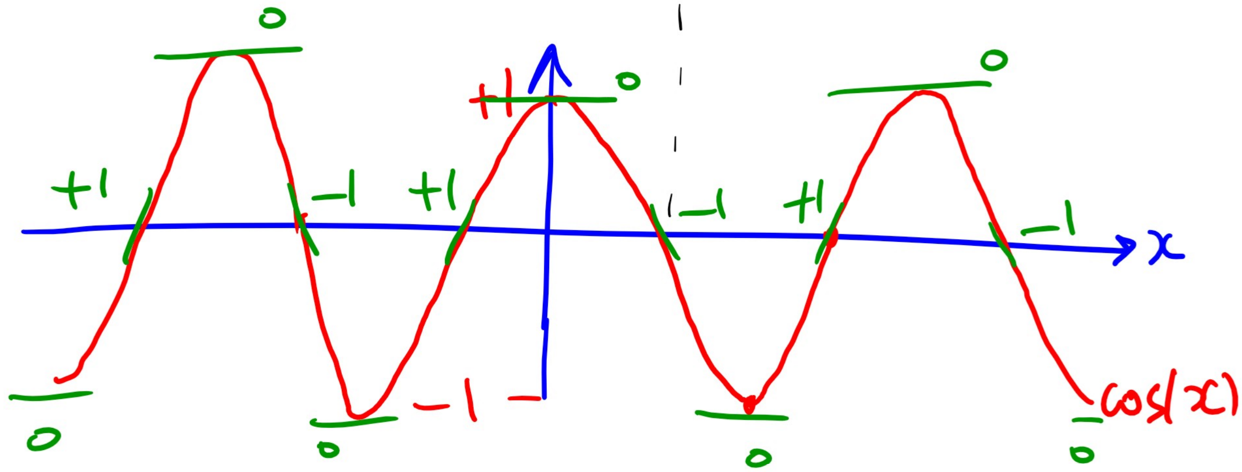
$= 1.e^x + x.e^x = (1+x)e^x$.

Trig. Functions



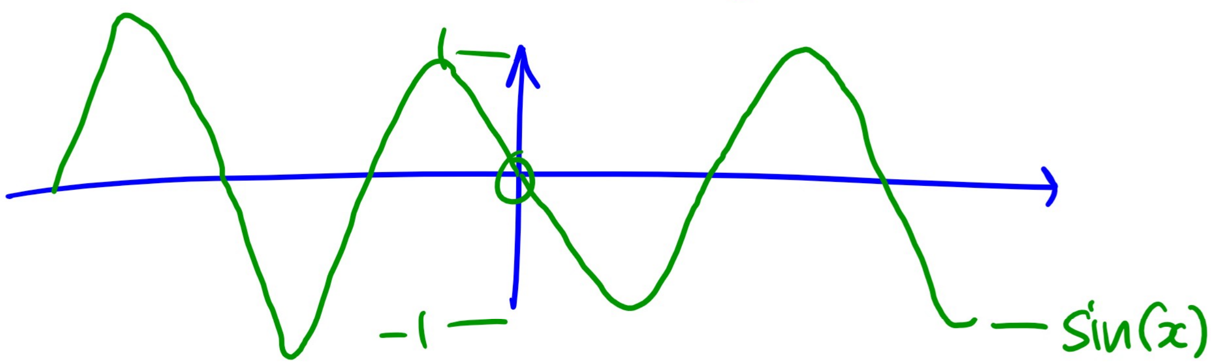
$(\sin x)'$

$= \cos x$



$(\cos x)'$

$= -\sin(x)$



$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{(\sin x)'\cos x - (\sin x)(\cos x)'}{\cos^2 x}$$

$$= \frac{\overbrace{\cancel{\cos^2 x + \sin^2 x}}^{=1}}{\cos^2 x} = \sec^2 x.$$

Example Let $y = \dfrac{\cos x}{1+\sin x}$. Find y'.

Solution

$$y' = \frac{(-\sin x)(1+\sin x) - (1+\sin x)'\cos x}{(1+\sin x)^2}$$

$$= (-\sin x - \sin^2 x - \cos^2 x)/(1+\sin x)^2$$

$$= \frac{\cancelto{-1}{-\sin x - 1}}{(1+\sin x)^{\cancel{2}}} = \frac{-1}{1+\sin x}$$

4.8 Newton's Method

↳ Used to approximate solutions to equations

Idea: Use tangent lines to "home in on" solution