1ZA3 (SECTION C01) - ENGINEERING MATHEMATICS I

Lecture 10

Last time Rules for derivatives when "combining" functions

e.g. PRODUCT RULE: $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
(A.K.A. LEIBNIZ RULE)

QUOTIENT RULE: $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

(where defined)

& Rules for differentiating basic functions

POWER RULE: $(x^n)' = nx^{n-1}$.

Works not only for positive integers n. (In fact for any n.)

Example Differentiate $f(x) = \frac{1}{\sqrt[5]{x^3}} - \frac{1}{\sqrt{x}}$

Solution Rewrite $f(x) = \frac{1}{x^{3/5}} - \frac{1}{x^{1/2}}$

$$= x^{-3/5} - x^{-1/2}$$

So $f'(x) = -\frac{3}{5}x^{-3/5-1} - \left(-\frac{1}{2}x^{-1/2-1}\right)$

$$= -\frac{3}{5}x^{-8/5} + \frac{1}{2}x^{-3/2}.$$

So in general a polynomial

$$f(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_d x^d$$

(where d is the degree of $f(x)$)

has $f'(x) = (0 +)\ c_1 + 2c_2 x + 3c_3 x^2 + \dots + d c_d x^{d-1}$.

Example Differentiate $f(x) = 3x^{100} + 2x^{36} + x^5$

Solution $f'(x) = 300x^{99} + 72x^{35} + 5x^4$.

Exponential Functions

$$f(x) = b^x \quad , b > 0 , b \neq 1.$$

$$f'(x) = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} = b^x \underbrace{\lim_{h \to 0} \left(\frac{b^h - 1}{h} \right)}_{f'(0)}$$

If it exists — & you are allowed to assume it does!

So $f'(x) = f'(0)\, b^x$

By definition of Euler's constant e, if $f(x) = e^x$ then $f'(0) = 1$, So $f'(x) = e^x$.

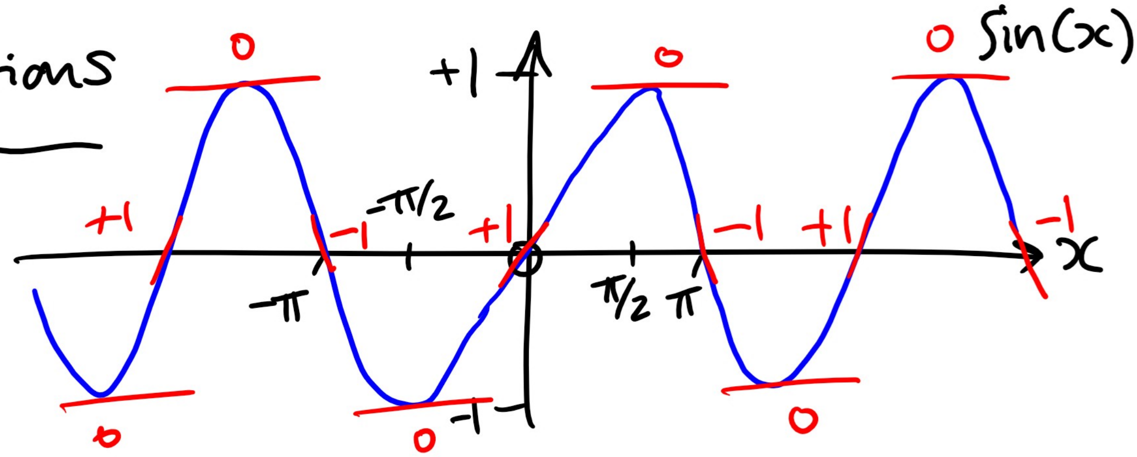
Example Differentiate $f(x) = xe^x$.

Solution $f'(x) = (x)'(e^x) + x(e^x)'$

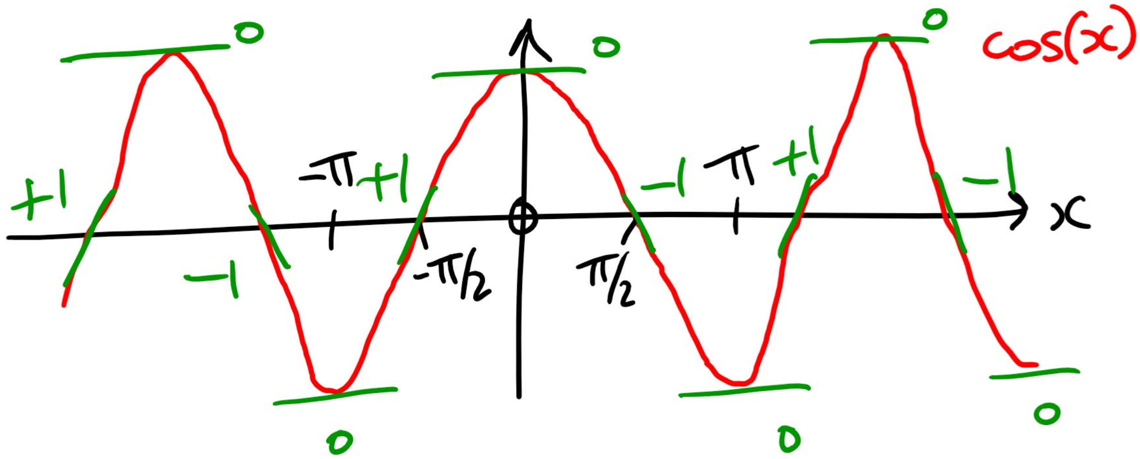
$$= 1.e^x + x.e^x$$

$$= (1+x)e^x.$$

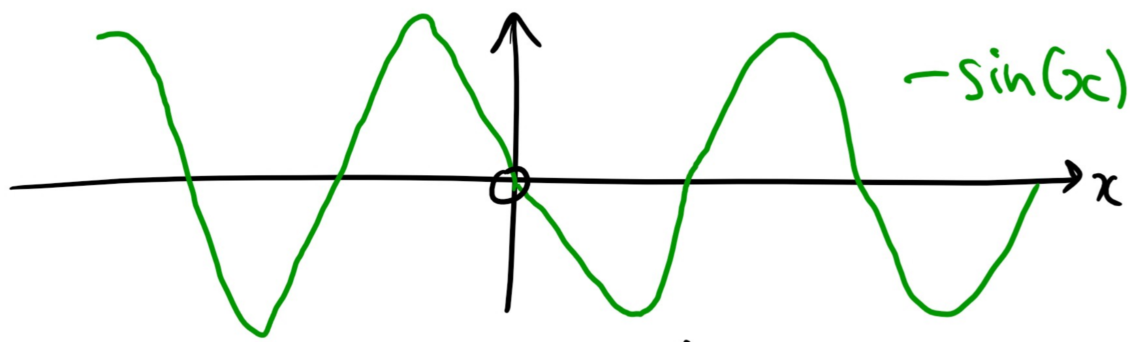
Trig. Functions



$(\sin(x))' = \cos(x)$



$(\cos(x))' = -\sin(x).$



$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'\cos x - (\cos x)'\sin x}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x.$$

<u>Example</u> Let $y = \dfrac{\cos x}{1+\sin x}$. Find y'.

<u>Solution</u>

$$y' = \frac{(-\sin x)(1+\sin x) - (1+\sin x)'\cos x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

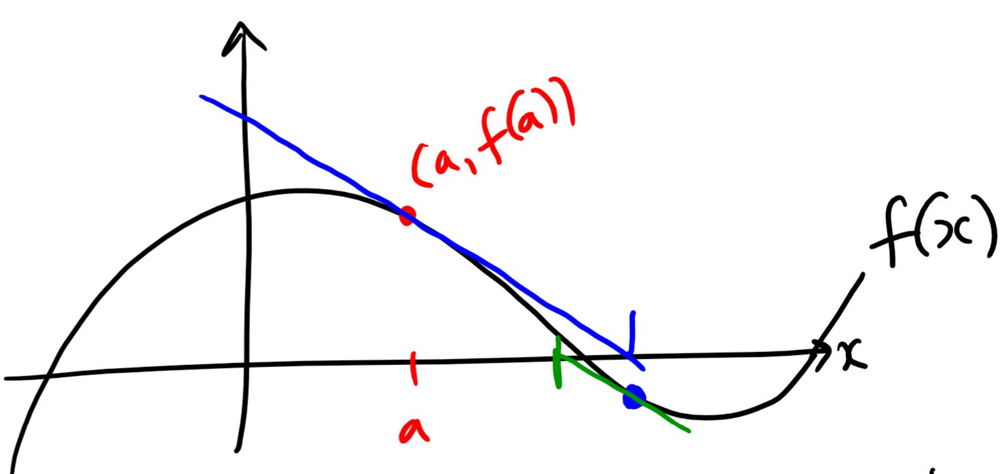
$$= \frac{\overset{-1}{\cancel{-\sin x - 1}}}{(1+\sin x)^{\cancel{2}}} = -\frac{1}{1+\sin x}.$$

4.8

Newton's Method

↳ Used to approximate solutions to equations $f(x) = 0$.

<u>Idea</u> Use tangent line at $(a, f(a))$ to "home in on" a solution.



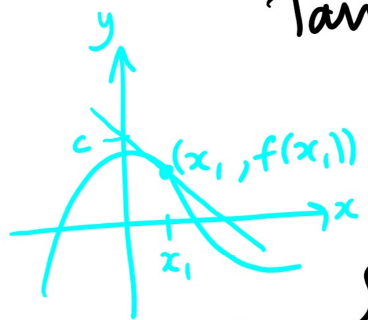
→ Take x-intercept of tangent line as the "new" x-value & look at <u>its</u> tangent line to $y = f(x)$, take <u>its</u> x-intercept as a new x-value, and so on...

More precisely: Set $x_1 = a$

Then $x_2 =$ x-intercept of tangent line at $(x_1, f(x_1))$ to $y = f(x)$.

How to find x_2?

Tangent line at $(x_1, f(x_1))$ is given by



$$y = f'(x_1)(x - x_1) + f(x_1)$$

So $$0 = f'(x_1)(x_2 - x_1) + f(x_1)$$

i.e. $$x_2 - x_1 = -\frac{f(x_1)}{f'(x_1)}$$

$$\Rightarrow \boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}$$

$y = mx + c$ where $m = f'(x_1)$

i.e. $y = f'(x_1)x + c$

& $f'(x_1) = \frac{\text{rise}}{\text{run}} = \frac{f(x_1) - c}{x_1}$

So, solving for c:

$c = f(x_1) - x_1 f'(x_1)$

Thus $y = f'(x_1)x + f(x_1) - x_1 f'(x_1)$

i.e. equation above.

If x_1 "reasonably" chosen then x_2 closer to a solution of $f(x) = 0$.

Now iterate with x_2 in place of x_1:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}.$$

And so on. In general

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Keep going until successive steps agree to some large (likely predetermined) number of decimal places.

<u>Remarks</u> 1. In the case that there is more than one solution, the initial guess a $(= x_1)$ determines which solution you home in on (if any).

2. For some curves/initial guesses this method doesn't work.