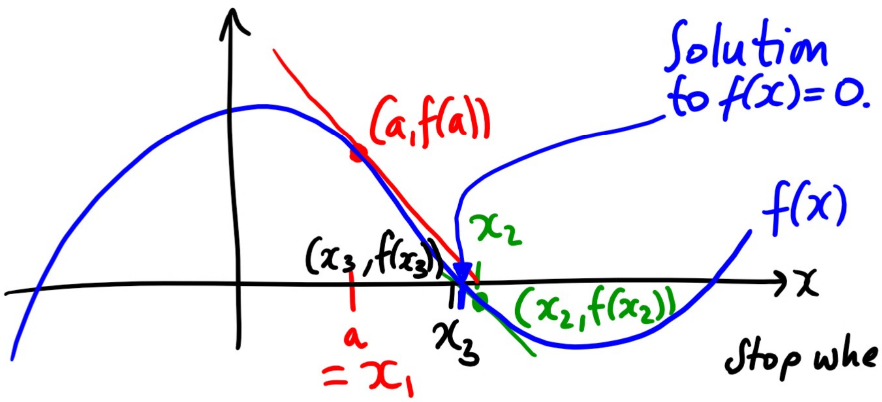
1ZA3 (SECTION C01) – ENGINEERING MATHEMATICS I

Lecture 11

Last time **NEWTON'S METHOD**

Use tangent lines to $y=f(x)$ to "home in on" solutions to $f(x)=0$.

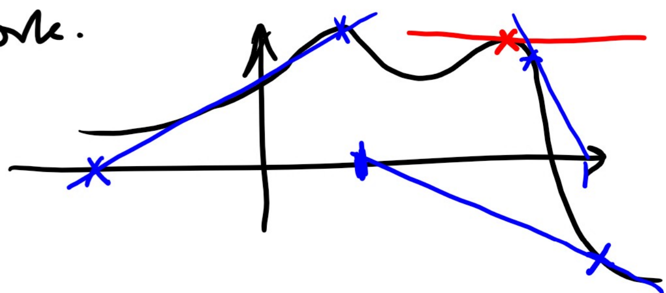


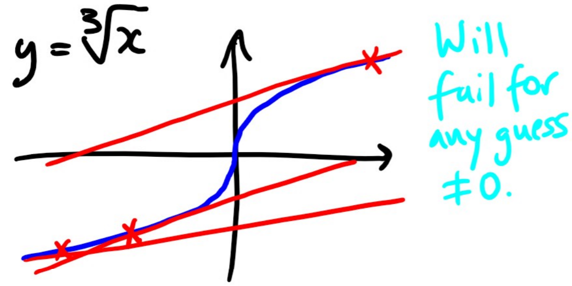
In general:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Stop when they agree to __ decimal places.

2. For some curves/initial guesses the method does not work.





Example Approximate a root of the equation $x^3 = x^2 + 2x - 1$ to 2 decimal places using Newton's method.

Solution First set up problem as $f(x)=0$.

$$\underbrace{x^3 - x^2 - 2x + 1}_{f(x)} = 0.$$

We need a sensible initial guess. How?

— try a few (whole number) values (whatever is easy to check)

— curve-sketching (use a computer?)

$f(0) = 1$

$f(1) = 1 - 1 - 2 + 1 = -1$

$f(2) = 8 - 4 - 4 + 1 = 1$

Want to choose an initial guess a so that $f(a)$ is close to 0 — here all of these good.

$f(x) = x^3 - x^2 - 2x + 1$

Say $x_1 = 1$.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{(-1)}{3-2-2} = 1 - \frac{(-1)}{(-1)} = 0$$

$f'(x) = 3x^2 - 2x - 2$

(Don't get confused! $x_2 = 0$ is our new x-value NOT a solution.)

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0 - \frac{f(0)}{f'(0)} = -\frac{1}{-2} = \frac{1}{2} = 0.5$$

$$x_4 = \frac{1}{2} - \frac{f(1/2)}{f'(1/2)} = \dots = \frac{4}{9} = \underline{0.444}\dots$$

$$x_5 = \frac{4}{9} - \frac{f(4/9)}{f'(4/9)} = \dots = \frac{4}{9} + \frac{1}{1674} = \underline{\underline{0.44504}}\dots$$

So our answer is $\underline{\underline{0.44}}$.

Example Approximate $\sqrt[7]{1000}$ to 8 decimal places using Newton's Method.

Solution Need equation $f(x)=0$.

$$x = \sqrt[7]{1000} \longrightarrow x^7 = 1000$$

$$\cancel{x - \sqrt[7]{1000} = 0} \qquad x^7 - 1000 = 0.$$

$$\underbrace{x^7 - 1000}_{f(x)}$$

This is an equation with the correct solution. But to use Newton's Method we would need to plug in $\sqrt[7]{1000}$ — exactly the # we want to approximate! (And how does your calculator know what $\sqrt[7]{1000}$ is? It probably uses something like Newton's Method!)

Let's say our initial guess is $x_1 = 3$ $[f(3) = 1187]$ $[f(2) = -872]$ — about 1000 away is "close" to 0 for this equation

— We're lazy, we try whole #s, we see that "about 1000 away from 0" is as good as we can hope to do.

Again we'll need $f'(x)$

$$= 7x^6.$$

$$x_2 = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{1187}{\cancel{7.3^6}\ 5103} = 2.76739\ldots$$

$$x_3 = 2.690087\ldots$$

..... our approximation is reached at step 6

as $x_5 = 2.68269579|9$
$\quad\ x_6 = \underline{2.68269579}|5$

These agree to 8 decimal places, so we stop.

Answer.

3.4 Chain Rule

If $h(x)$ is differentiable at x and $g(x)$ is differentiable at $h(x)$

then $f(x) = (g \circ h)(x) = g(h(x))$ is differentiable at x and $f'(x) = g'(h(x)).\,h'(x)$.

Alternatively if $y = g(u)$ and $u = h(x)$
(so $y = f(x) = g(h(x))$)

then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ ← Not quotients! But pretend they are & imagine cancelling.

<u>Example</u> $y = \underbrace{e}_{\text{outside}}^{x^2}$ inside .

<u>Solution</u> $y = g(u) = e^u$
$u = h(x) = x^2$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$
$$= e^u.(2x)$$
$$= (2x)e^{x^2}.$$

In general $\boxed{e^{h(x)} = h'(x)e^{h(x)}}$

<u>Example</u> Let $f(x) = \underbrace{\tan}_{\text{outer}}(\underbrace{5x-1}_{\text{inner}})$. $u = h(x) = 5x-1$

$g(u) = \tan(u)$

Solution
$$f'(x) = g'(h(x)).h'(x)$$
$$= \sec^2(h(x)) . h'(x)$$
$$= 5\sec^2(5x-1).$$

Example Let $f(x) = b^x$. What is $f'(x)$?
$b > 0$
$b \neq 1$

Solution We already saw that $f'(x) = f'(0).b^x$
(And, indeed, if $b=e$, then $f'(x) = e^x$.)

But now we can be cleverer:
$$f(x) = b^x = e^{\ln(b^x)} = e^{x\ln b}$$
Now we have a chain of functions that we DO know how to differentiate: e^u, const×x

So $f'(x) = (x\ln b)' e^{x\ln b}$ (using above)
$$= (\ln b)\underbrace{e^{x\ln b}}_{b^x} = (\ln b) b^x.$$
So yes, for $f(x) = b^x$, $f'(0) = \ln b$.

Example If $f(x) = g(h(x))$ find $f'(-1)$
if $h(-1)=2$, $h'(-1)=3$, $g'(2)=4$.

Solution
$$f'(x) = g'(h(x))\,h'(x)$$
So $f'(-1) = g'(h(-1))\,h'(-1)$
$$= g'(2).3$$
$$= 4.3 = 12.$$