1ZA3 (SECTION C01) - ENGINEERING MATHEMATICS I

Lecture 12

Last time

CHAIN RULE

If $f(x) = g(h(x))$ and everything is differentiable then

$$f'(x) = g'(h(x)) \cdot h'(x).$$

If we set $y = f(x) = g(u)$, with $u = h(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

Warm-Up If $f(x)$ is differentiable & n is a real # find $\frac{d}{dx}\left((f(x))^n\right)$.

Solution set $g(u) = u^n$, $u = f(x)$

Then $\frac{d}{dx}\left(f(x)^n\right) = g'(u) \cdot f'(x)$

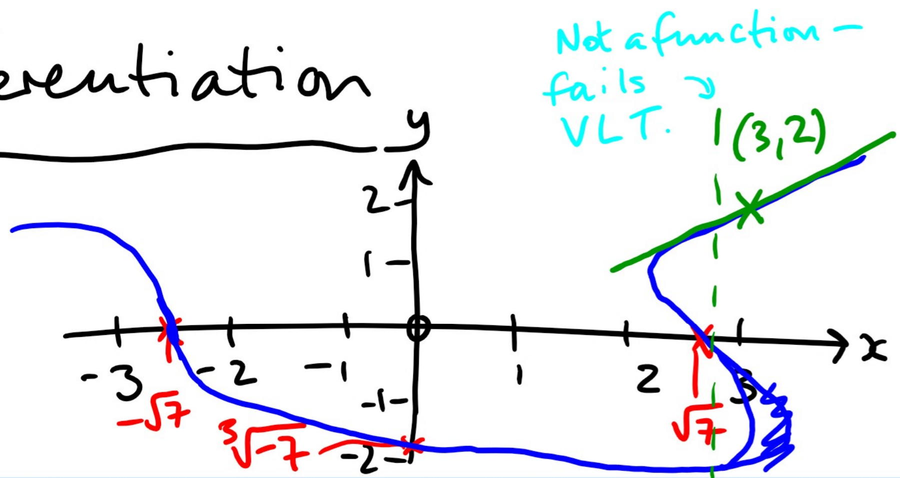
$= n(f(x))^{n-1} \cdot f'(x).$

Power Rule + Chain Rule

3.5 Implicit Differentiation

Consider

$$x^2 + xy = 7 + y^3$$



This curve is NOT the graph of a function BUT at each point on the curve, there IS a tangent line.

Problem: What is the equation of the tangent line to the curve at the point $(3,2)$? $\leftarrow$ $9+6=7+8$

So this really is a point on the curve!

Solution: One idea: solve for y explicitly in terms of x — essentially impossible here

Instead: Use implicit derivative

Idea: The equation $x^2+xy=7+y^3$ defines y implicitly as several functions of x.

So differentiate both sides of the equation with respect to x and then solve for y'.

$'$ means here w.r.t. x

$$(x^2+xy)' = (7+y^3)'$$

$$2x + x'y + xy' = 0 + (y^3)'$$

$$2x + y + xy' = 3y^2 \cdot y'$$

$\leftarrow$ Power Rule + Chain Rule

(Think "y is a function of x" $y=y(x)$ or replace: $y=f(x)$.)

Solve for y':

$$xy' - 3y^2y' = -2x-y$$

$$y'(x-3y^2) = -(2x+y)$$

$$y' = -\frac{(2x+y)}{(x-3y^2)}$$

So at the point $(x,y) = (3,2)$ we have

$$y' = \frac{-(2\cdot 3 + 2)}{(3 - 3\cdot 4)} = \frac{-8}{-9} = \frac{8}{9}.$$

Then find equation for the tangent line at $(3,2)$ in the usual way

$$y = y'(x-3) + 2 = \frac{8}{9}x - \frac{2}{3}.$$

Inverse Trig. Functions

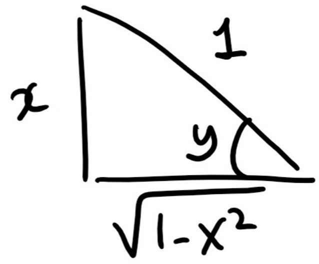
Example Let $y = \arcsin(x)$. Find $\frac{dy}{dx}$.

Solution $\sin(y) = x \qquad y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Use implicit differentiation:

$$(\sin(y))' = x'$$

$$y'\cos(y) = 1 \Rightarrow y' = \frac{1}{\cos(y)}$$



$$= \frac{1}{\cos(\arcsin(x))}$$

$$\boxed{y' = \frac{1}{\sqrt{1-x^2}}}$$

Similarly: $\frac{d}{dx}(\arccos(x)) = -\frac{1}{\sqrt{1-x^2}}$

$$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2} \qquad \left[\begin{array}{l} \tan(y) = x \\ \sec^2(y) \cdot y' = 1 \\ \vdots \end{array} \right]$$

Example Find $f'(2\sqrt{3})$ when $f(x) = x \tan^{-1}\left(\frac{x}{2}\right)$.

Solution To follow

More generally : **Derivatives of inverse functions**

Suppose f is 1–1 and f is differentiable and f^{-1} is differentiable.

What is $(f^{-1})'$?

We set $y = f^{-1}(x)$ so $x = f(y)$.

Differentiate implicitly: $1 = f'(y) \cdot y'$

$$\text{So } y' = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}.$$

$$\text{So } \boxed{\left(f^{-1}(x)\right)' = \frac{1}{f'(f^{-1}(x))}.}$$

Example $f(x) = \sin(x)$. Find $(f^{-1}(x))'$

Solution $\left(f^{-1}(x)\right)' = \dfrac{1}{\cos(\arcsin(x))}$ -- simplify as above.

Example Let $f(x) = \sqrt{x} + x^2 + 1$, $x \geq 0$

Find the slope of the tangent line to $y = f^{-1}(x)$ at the point $(3, 1)$

$(f(1) = 3$, so $f^{-1}(3) = 1)$

Solution $\dfrac{dy}{dx} = \dfrac{1}{f'(f^{-1}(x))}$

So $\left.\dfrac{dy}{dx}\right|_{x=3} = \dfrac{1}{f'(f^{-1}(3))} = \dfrac{1}{f'(1)} = \dfrac{1}{(5/2)} = \dfrac{2}{5}$.

evaluate at $x=3$

What is $f'(x)$?

$= \frac{1}{2}x^{-1/2} + 2x$

3.6 Derivatives of Log. Functions

For $y = \log_b x$, $b > 0$, $b \neq 1$

we have $b^y = x$

so by implicit differentiation wrt x,

$$y' = \frac{1}{\underbrace{(\ln b)\, b^y}_{\text{derivative of } b^y \text{ wrt } y}} = \frac{1}{(\ln b)\, x}.$$

— see "Derivatives of Exponential Functions".

If $b = e$ & $f(x) = \ln x$, we get $f'(x) = \dfrac{1}{x}$.

Example Let $f(x) = \ln(\sin^2 x)$. Find $f'(x)$.

$y = \ln(u)$; $u = v^2$; $v = \sin x$.

Solution
$$f'(x) = \frac{1}{\sin^{\cancel{2}} x} \cdot (2\cancel{\sin x}) \cos x$$
$$= 2\frac{\cos x}{\sin x} = 2\cot x.$$

In general, by Chain Rule, $\frac{d}{dx}(\ln(h(x)))$
$$= \boxed{\frac{h'(x)}{h(x)}}$$

This is called the "logarithmic derivative of $h(x)$", but you don't need to know that. :)