1ZA3 (SECTION C01) - ENGINEERING MATHEMATICS I

Last time

IMPLICIT DIFFERENTIATION

- We can differentiate an implicit definition of y in terms of x.
- e.g. for INVERSE FUNCTIONS $\longrightarrow$ $\left(f^{-1}(x)\right)' = \dfrac{1}{f'(f^{-1}(x))}$.
- Special case: $f(x) = \ln(x)$; $f'(x) = \dfrac{1}{x}$.

$\hookrightarrow$ so $\left(\ln(g(x))\right)' = g'(x)/g(x)$.

Logarithmic Differentiation

If $f(x)$ - hard to differentiate

but $\ln f(x)$ - easy (or easier) to differentiate

We can then say if $y = f(x)$

$$\ln y = \ln f(x)$$

(implicit diff.) $\Rightarrow$ $$\frac{y'}{y} = \frac{d}{dx}\left(\ln f(x)\right)$$

$$\Rightarrow y' = y \cdot \frac{d}{dx}\left(\ln f(x)\right) = f(x) \cdot \frac{d}{dx}\left(\ln f(x)\right)$$

Example $y = \sqrt{\dfrac{x+1}{x^3-1}}$. Find y'.

Solution could use quotient rule - unpleasant.

Instead : take logs & use implicit differentiation

$$\ln y = \ln\left(\frac{x+1}{x^3-1}\right)^{1/2}$$

$$\ln y = \frac{1}{2}\ln\left(\frac{x+1}{x^3-1}\right)$$

Much nicer to differentiate!

$$\ln y = \overbrace{\frac{1}{2}\left(\ln(x+1) - \ln(x^3-1)\right)}$$

$$\rightsquigarrow \quad \frac{y'}{y} = \frac{1}{2}\left(\frac{\overbrace{1}^{=(x+1)'}}{x+1} - \frac{\overbrace{3x^2}^{=(x^3-1)'}}{x^3-1}\right)$$

$$y' = \frac{1}{2}\left(\frac{1}{x+1} - \frac{3x^2}{x^3-1}\right)\left(\frac{x+1}{x^3-1}\right)^{1/2}$$

$\rightsquigarrow$ simplify. $-\frac{1}{2}\left(\frac{2x^3+3x^2+1}{(x+1)^{1/2}(x^3-1)^{3/2}}\right)$

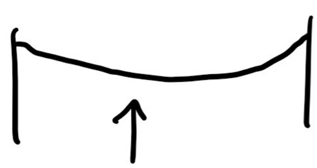
3.11 Hyperbolic Functions

— Family of functions related to trig. functions (also to exp. functions)

— Arise in many applications

e.g. — fluid dynamics (wave motion)

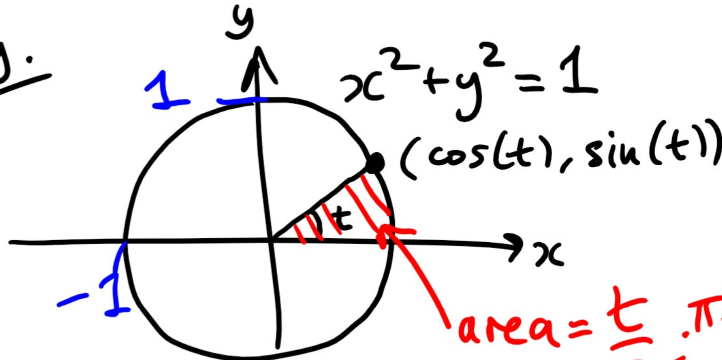
— engineering (hanging wire)



catenary = hyperbolic sine

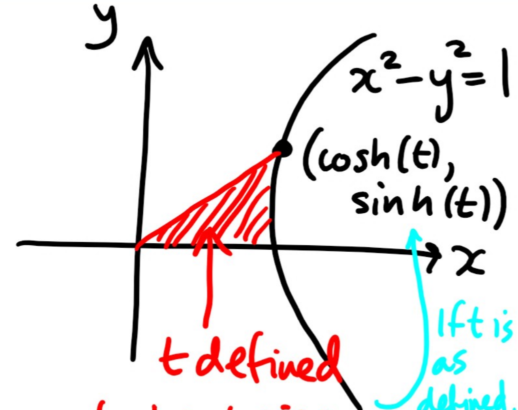
"Hyperbolic functions are to hyperbolas as Trig. functions are to circles"!

Trig.



area $= \frac{t}{2\pi} \cdot \pi = \frac{t}{2}$

Hyp.



t defined to be twice this area

If t is as defined, then the point on the hyperbola has these coords.

Definition of hyperbolic functions

"sinch" (or "shine") $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$

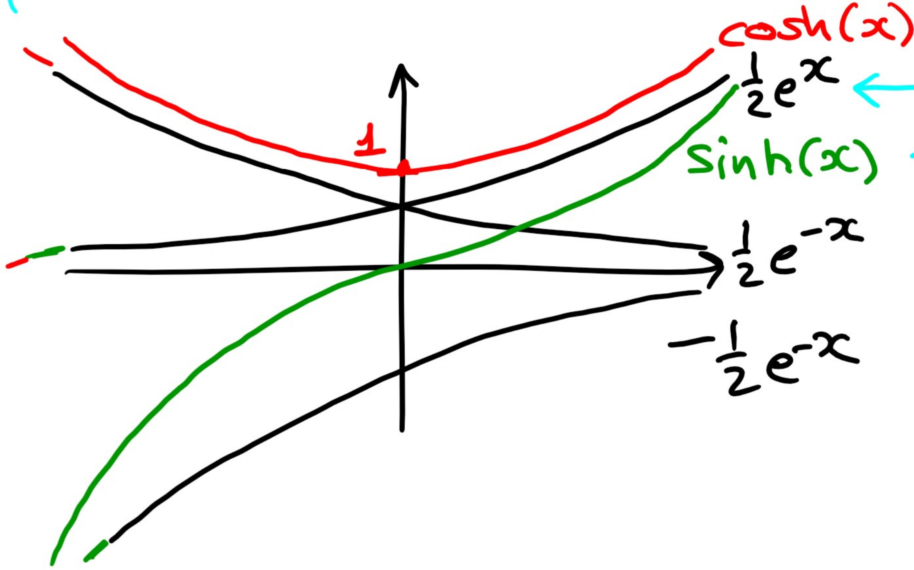
hyperbolic cosecant ↳ $\mathrm{csch}(x) = \dfrac{1}{\sinh(x)}$

"cosh" $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$

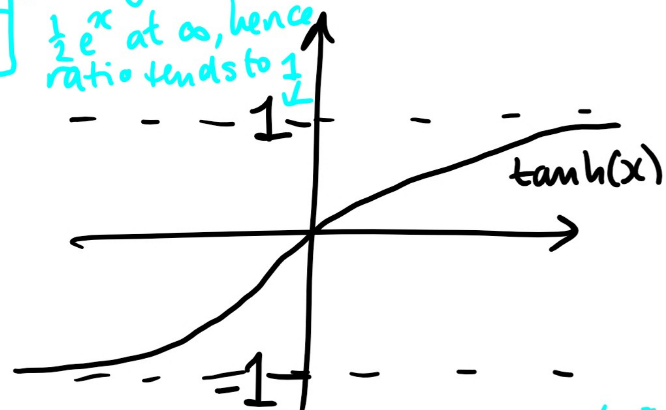
$\mathrm{sech}(x) = \dfrac{1}{\cosh(x)}$

"tanch" (or "than") $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$\coth(x) = \dfrac{1}{\tanh(x)}$



both get close to $\frac{1}{2}e^x$ at ∞, hence ratio tends to 1



Similarly, sinh(x) gets close to $-\frac{1}{2}e^{-x}$ & cosh(x) " " " $\frac{1}{2}e^{-x}$ at $-\infty$.

Identities

$-\sinh(x) = \sinh(-x)$ (odd)

$\cosh(x) = \cosh(-x)$ (even)

$-\tanh(x) = \tanh(-x)$ (odd)

$$\cosh^2(x) = \left(\frac{e^x + e^{-x}}{2}\right)^2 = \frac{e^{2x} + 2 + e^{-2x}}{4}$$

$$\sinh^2(x) = \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} - 2 + e^{-2x}}{4}$$

$$\cosh^2(x) - \sinh^2(x) = \frac{2-(-2)}{4} = \frac{4}{4} = 1$$

Also $\cosh^2(x) + \sinh^2(x) = \frac{\cancel{2}e^{2x} + \cancel{2}e^{-2x}}{\cancel{4}\,2} = \cosh(2x)$

Divide by $\cosh^2(x)$: $1 - \tanh^2(x) = \operatorname{sech}^2(x)$

or by $\sinh^2(x)$: $\coth^2(x) - 1 = \operatorname{csch}^2(x)$

Also seen from here with $y=x$.

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$$
$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$

Exercises

Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh(x)) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh(x)$$

$$\frac{d}{dx}(\cosh(x)) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh(x)$$

$$\frac{d}{dx}(\tanh(x)) = \frac{d}{dx}\left(\frac{\sinh(x)}{\cosh(x)}\right) = \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} = \operatorname{sech}^2(x)$$

Example Let $f(x) = \ln(\sinh(x))$. Find $f'(x)$.

Solution $f'(x) = \frac{(\sinh(x))'}{\sinh(x)} = \frac{\cosh(x)}{\sinh(x)} = \coth(x)$.

Inverses of Hyperbolic Functions

Which graphs pass the HLT?

$\sinh(x)$, $\tanh(x)$

✓ for all $x \in \mathbb{R}$ ✓ for all $x \in \mathbb{R}$

$\cosh(x)$ for $x \geq 0$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

set $y = \dfrac{e^x - e^{-x}}{2} \Rightarrow 2y = e^x - e^{-x} = e^x - \dfrac{1}{e^x}$

$$\Rightarrow 2ye^x = (e^x)^2 - 1$$

$$\Rightarrow (e^x)^2 - 2y e^x - 1 = 0$$

← quadratic in e^x

$$\rightarrow \quad e^x = \frac{2y \pm \sqrt{4y^2+4}}{2} = \frac{\cancel{2}y}{\cancel{2}} \pm \frac{\sqrt{\cancel{4}y^2+\cancel{4}1}}{\sqrt{\cancel{4}}}$$

$$= y \pm \sqrt{y^2+1}$$

-ve $\sqrt{}$ is -ve soln. & e^x cannot be -ve

$y^2+1 > y^2$ so $\sqrt{y^2+1} > \sqrt{y^2} = y$

$$\Rightarrow e^x = y + \sqrt{y^2+1}$$

$$\Rightarrow x = \ln\left(y + \sqrt{y^2+1}\right)$$

$$\rightsquigarrow y = \ln\left(x + \sqrt{x^2+1}\right)$$

(Swapping roles of x and y.)

$$y = \sinh^{-1}(x).$$