1ZA3 (SECTION C01) – ENGINEERING MATHEMATICS I

Lecture 14

Last time HYPERBOLIC FUNCTIONS

$$\sinh(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right) \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\cosh(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

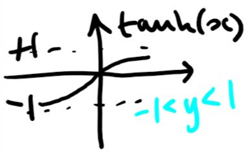
$\text{sech}(x)$, $\text{csc}(x)$, $\coth(x)$
$= 1/\cosh(x)$ $= 1/\sinh(x)$ $= 1/\tanh(x)$

Derivatives: $(\sinh(x))' = \cosh(x)$, $(\cosh(x))' = \sinh(x)$

Inverses: $\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right).$

TBC...

Similarly: $\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2-1}\right)$ for $x \geq 1$



$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \text{ for } -1 < x < 1$$

Derivatives of Inverses

Can differentiate above formulas directly or use

$$\left(f^{-1}(x)\right)' = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

e.g. $\left(\sinh^{-1}(x)\right)' = \dfrac{1}{\cosh\left(\sinh^{-1}(x)\right)}$

or $\left(\sinh^{-1}(x)\right)' = \left(\ln\left(x + \sqrt{x^2+1}\right)\right)'$

$$= \frac{\left(x + (x^2+1)^{1/2}\right)'}{x + (x^2+1)^{1/2}}$$

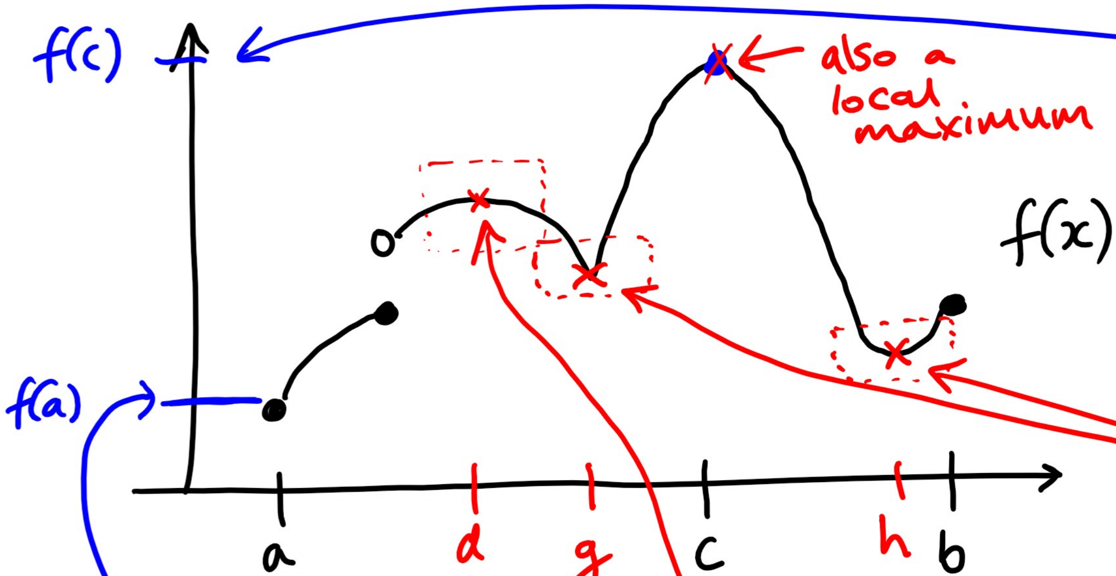
$$= \frac{1 + \cancel{\tfrac{1}{2}}(x^2+1)^{-1/2}(\cancel{2}x)}{x + (x^2+1)^{1/2}}$$

$$= \frac{\cancel{(x^2+1)^{1/2} + x}}{(x^2+1)^{1/2}\left(\cancel{x + (x^2+1)^{1/2}}\right)} = \frac{1}{\sqrt{x^2+1}}$$

Also $\left(\cosh^{-1}(x)\right)' = \dfrac{1}{\sqrt{x^2-1}}$, $\left(\tanh^{-1}(x)\right)' = \dfrac{1}{1-x^2}$.

"Using derivatives to solve problems"

4.1 Maximum & Minimum Values



largest value of $f(x)$ on $[a,b]$ = absolute maximum of $f(x)$ on $[a,b]$

i.e. $f(c) \geq f(x)$ for all x in $[a,b]$

local minima of $f(x)$:

$f(g) \leq f(x)$ for all x near g

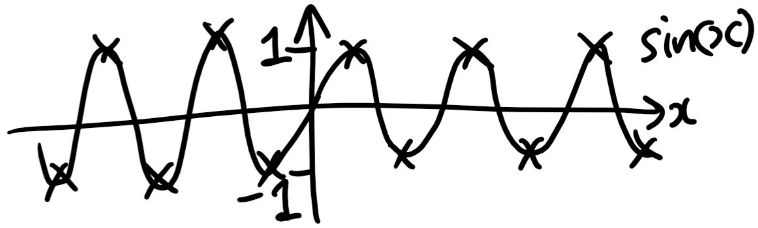
$f(h) \leq f(x)$ for all x near h

Smallest value of $f(x)$ on $[a,b]$ = absolute minimum of $f(x)$ on $[a,b]$

i.e. $f(a) \leq f(x)$ for all x in $[a,b]$

local maximum of $f(x)$:

$f(d) \geq f(x)$ for all x near d



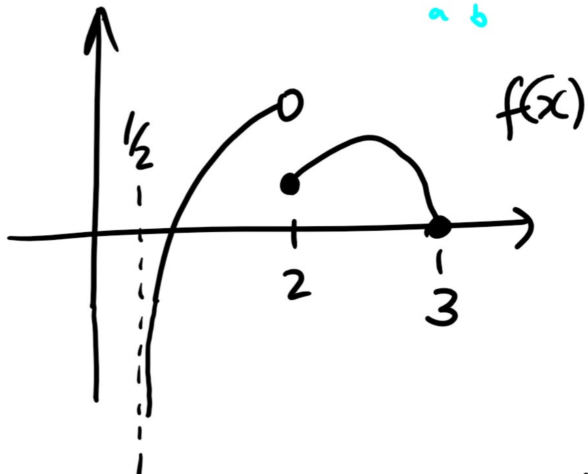
← absolute maximum

or

← absolute minimum

are values that $f(x)$ can take & may be "attained" at more than one x-value.

For a constant function: $f(x) = k$, say, k is the abs. max. & abs. min. of $f(x)$ and both are attained everywhere.



No absolute minimum

$\lim_{x \to \frac{1}{2}^+} f(x) = -\infty$

No absolute maximum

You want it to be $\lim_{x \to 2^-} f(x)$ but this is a value that does not occur as $f(c)$ for any x-value c (and in particular does not equal $f(2)$).

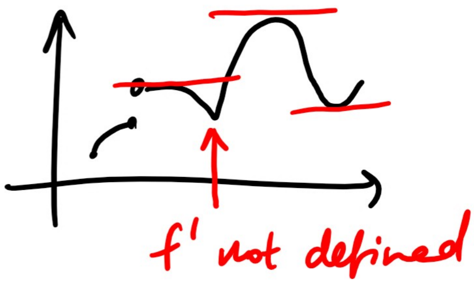
How can we guarantee that we have absolute max. & min. & how do we then find them?

Extreme Value Theorem

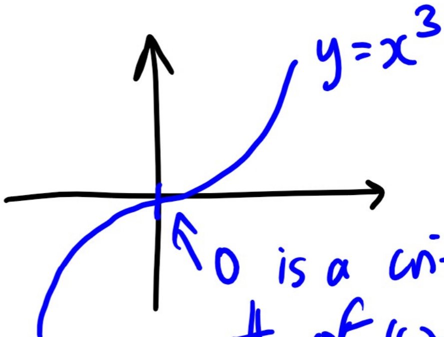
If $f(x)$ is continuous on a closed interval $[a, b]$ then f attains an absolute maximum and an absolute minimum on $[a, b]$.

But how to find them?

Fermat's Theorem If $f(x)$ has a local max. or local min. at c then either $f'(x)$ is not defined at $x = c$ or $f'(c) = 0$.



Definition A critical number of a function $f(x)$ is any c in dom(f) for which $f'(x)$ is not defined at $x = c$ or $f'(c) = 0$.



0 is a critical # of $f(x) = x^3$ but $f(0)$ NOT a local max./min.

WARNING
Not every critical value gives a local max/min

But Fermat's Theorem tells us if we're looking for local max./min. we can restrict our search to critical numbers.

Closed Interval Method

Notice abs. max. (or min.) of a continuous function are either at local max. (or min.) or at endpoints of interval.

Text book says these are NOT the same thing.

Strategy for searching for abs. max./min. of $f(x)$ continuous on $[a,b]$:

① Find $f(c)$ for every critical # c in $[a,b]$

② Find $f(a)$ and $f(b)$.

③ The biggest of all values in ① and ② is the absolute max. and the smallest is the absolute minimum