1ZA3 (SECTION C01) - ENGINEERING MATHEMATICS I

Lecture 15

Last time

MAXIMUM & MINIMUM VALUES

Extreme Value Theorem tells us : if $f(x)$ continuous on $[a,b]$ closed
then $f(x)$ attains an absolute maximum & minimum.

Where? Either at local maximum/minimum or endpoints

So we check $f(c)$ for critical values c and $f(a), f(b)$.

$f'(c)=0$ or $f'(c)$ not defined

Example Find absolute max. & min. values of

$$f(x) = x^{2/3}(6-x) \text{ on } [-1,6].$$

Solution (We'll take on trust that $f(x)$ is continuous.)

① Find critical numbers of $f(x)$:

$$f'(x) = \left(6x^{2/3} - x^{5/3}\right)' = 6\left(\tfrac{2}{3}\right)x^{-1/3} - \left(\tfrac{5}{3}\right)x^{2/3}$$

$$= 4x^{-1/3} - \frac{5}{3}x^{2/3}$$

$$= \frac{4 - 5/3\,x}{x^{1/3}}$$

$0 = f'(x)$ when $4 - 5/3\,x = 0$

i.e. $x = 12/5$

and $f'(x)$ undefined - when $x^{1/3} = 0$ i.e. $x = 0$

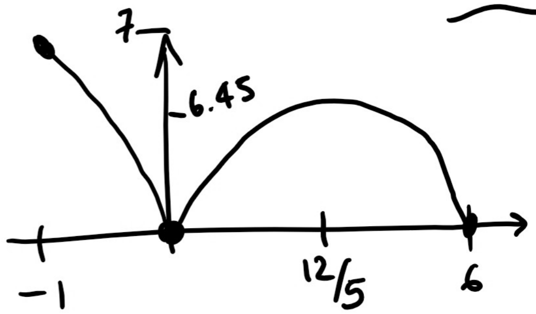
Now find $f(12/5) = \left(\frac{12}{5}\right)^{2/3}\left(6-\frac{12}{5}\right) \approx \boxed{6.45.}$

$f(0) = 0^{2/3}(6-0) = \boxed{0}$

② $f(-1) = (-1)^{2/3}(6-(-1)) = \boxed{7}$

$f(6) = 6^{2/3}(6-6) = \boxed{0}$

③ Compare: absolute max. $= 7$ (at $x=-1$)
absolute min. $= 0$ (at $x=0$ and $x=6$).

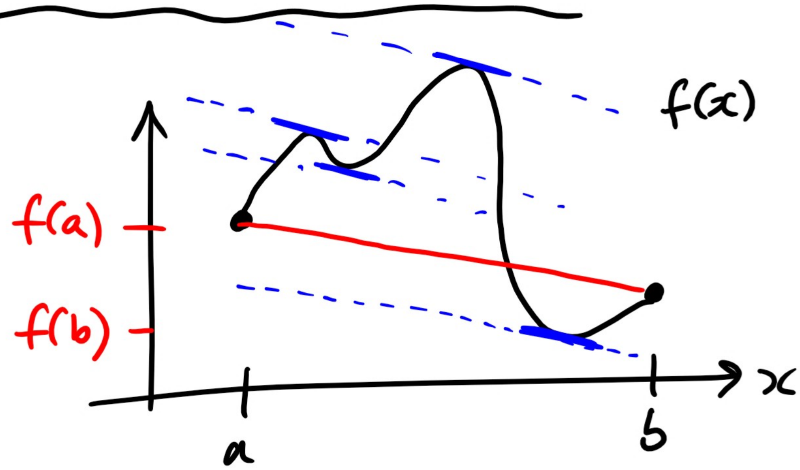


Exercise Find the critical numbers of $f(x) = 2\sin(x) + \cos(2x)$ on $[0, 2\pi]$.

UP TO HERE FOR MIDTERM 1.

4.2 The Mean Value Theorem

Pictorially



"Average rate of change of $f(x)$ on $[a,b]$" $= \dfrac{f(b)-f(a)}{b-a}$

Mean Value Theorem

Let $f(x)$ be a function with ① $f(x)$ continuous on $[a,b]$
② $f(x)$ differentiable on (a,b)

Then there is some $c \in [a,b]$ with $f'(c) = \dfrac{f(b) - f(a)}{b-a}$.

<u>Rolle's Theorem</u> If $f(a) = f(b)$, then if $f(x)$ meets conditions ① and ② from the MVT then there is some $c \in [a,b]$ with $f'(c) = 0$.

<u>Example</u> Find c satisfying the conclusion of MVT for

$$f(x) = \frac{x}{x+3} \quad \text{on} \quad [1,2].$$

<u>Solution</u> i.e. want c with $f'(c) = \dfrac{f(2) - f(1)}{2-1}$

$$= 2/5 - 1/4 = 3/20.$$

Now $f'(x) = \left(x(x+3)^{-1}\right)'$

$$= (x+3)^{-1} - x(x+3)^{-2}(1) = \frac{1}{x+3} - \frac{x}{(x+3)^2}$$

$$= \frac{x+3-x}{(x+3)^2} = \frac{3}{(x+3)^2}.$$

So now solve for c in $\dfrac{3}{20} = \dfrac{3}{(c+3)^2}$

i.e. $20 = (c+3)^2$

$\pm\sqrt{20} = c+3$

$c = -3 \pm \sqrt{20}$ So c value (in $[1,2]$)

is $c = \sqrt{20} - 3 \approx \underline{\underline{1.47}}.$

4.3 Derivatives & Graph Shape

$f'(a)$ – instantaneous rate of change of $f(x)$ at $x = a$

If $f'(a) > 0$, does this mean $f(x)$ is actually increasing?

Yes!

Scaling up:

Increasing / Decreasing Test

(a) If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on that interval.

Let's call our interval $[a, b]$.

(e.g. Since somewhere c in $[a, b]$ we have

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

, then $f'(x) > 0$ everywhere means $f'(c) > 0$ so $f(b) > f(a)$

and $f'(x) < 0$ everywhere means $f'(c) < 0$ so $f(b) < f(a)$.)

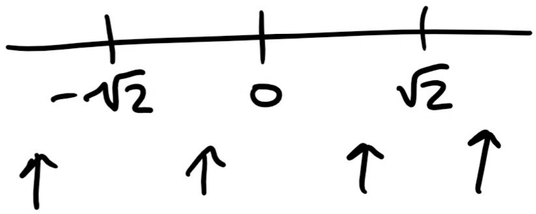
This tells us that the only x-values where $f(x)$ could potentially switch from increasing to decreasing (or decreasing to increasing) are when $f'(x) = 0$.

Example Find where $f(x) = x^6 - 3x^4 + 3$ is increasing and decreasing.

Solution Separate the real line into intervals using x-values where $f'(x)=0$:

$$0 = f'(x) = 6x^5 - 12x^3 = 6x^3(x^2-2)$$

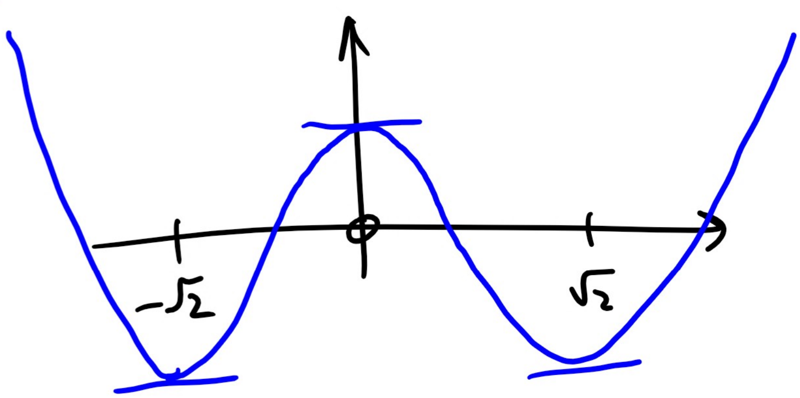
$x = 0 \qquad x = \pm\sqrt{2}$



Four intervals

	$f'(x) =$	$6x^3$	(x^2-2)	dec/inc
$(-\infty, -\sqrt{2})$	$-$	$-$	$+$	dec
$(-\sqrt{2}, 0)$	$+$	$-$	$-$	inc
$(0, \sqrt{2})$	$-$	$+$	$-$	dec
$(\sqrt{2}, \infty)$	$+$	$+$	$+$	inc

In fact the graph looks like this:



← Notice how the solutions to $f(x)=0$ are irrelevant to the question of whether or not f is increasing or decreasing.