1ZA3 (SECTION C01) - ENGINEERING MATHEMATICS I

Lecture 17

Last time

INDETERMINATE FORMS

We are interested in finding $\lim_{x\to a} \frac{f(x)}{g(x)}$ where

- $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ ← indeterminate form of type $\frac{0}{0}$

or

- $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$ ← indeterminate form of type $\frac{\infty}{\infty}$

can be different i.e. $\frac{-\infty}{\infty}$ or $\frac{\infty}{-\infty}$

Let's look $\lim_{x\to 0} \frac{\sin(x)}{3(e^x-1)}$. Zoom in on $x=0$.

Look at tangent lines at $x=0$:

↳ If $f(0)=0$, the equation of the tangent line to $f(x)$ at $x=0$ is $y=f'(0).x$.

$$\sin x \approx (\sin(x))'(0).x = \cos(0)x = x$$

$$3(e^x-1) \approx (3(e^x-1))'(0).x = 3e^0.x = 3x$$

So $\frac{\sin x}{3(e^x-1)} \approx \frac{x}{3x} = \frac{1}{3}$ at $x=0$.

So we expect $\lim_{x\to 0}\left(\frac{\sin x}{3(e^x-1)}\right) = \frac{1}{3}$.

L'Hospital's Rule

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

OR if $\lim_{x\to a} f(x) = \pm\infty$ AND $\lim_{x\to a} g(x) = \pm\infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$ ← If this limit exists or is $\pm\infty$

Here a could be any real # or $\pm\infty$.

Also works for one-sided limits (replacing a with a^+ or a^-)

<u>Example</u> Find $\lim_{x\to 0} \frac{e^{2x}-1}{x}$.

<u>Solution</u> $\lim_{x\to 0} e^{2x}-1 = \lim_{x\to 0} x = 0$ so

L'Hospital applies and $\lim_{x\to 0}\left(\frac{e^{2x}-1}{x}\right) = \lim_{x\to 0} \frac{2e^{2x}}{1}$

↑ $0/0$

$= 2$.

<u>Example</u> Find $\lim_{x\to\infty}\left(\frac{x-e^x}{x^2}\right)$.

Properly covered by "indeterminate differences" — see next lecture!

<u>Solution</u> $\lim_{x\to\infty}(x-e^x) = -\infty$, $\lim_{x\to\infty} x^2 = \infty$

So L'Hospital applies $\left(\frac{\infty}{\infty}\right)$:

$$\lim_{x\to\infty}\left(\frac{x-e^x}{x^2}\right) = \lim_{x\to\infty}\left(\frac{1-e^x}{2x}\right)$$ ← indeterminate form $\frac{\infty}{\infty}$

So L'Hospital applies!

$$= \lim_{x \to \infty} \left(\frac{-e^x}{2} \right) = -\infty .$$

Example Find $\lim_{x \to \frac{\pi}{2}} \left(\frac{1 - \sin(x)}{\csc(x)} \right)$.

Solution $\lim_{x \to \frac{\pi}{2}} (1 - \sin(x)) = 0$ & $\lim_{x \to \frac{\pi}{2}} \csc(x)$

$$= \lim_{x \to \frac{\pi}{2}} \frac{1}{\sin(x)} = 1$$

So L'Hospital DOES NOT APPLY HERE!!!

But $\lim_{x \to \pi/2} \left(\frac{1 - \sin(x)}{\csc(x)} \right) = \frac{0}{1} = 0.$

Example Find $\lim_{x \to -\infty} x . e^x$.

Solution $\lim_{x \to -\infty} x e^x$ has the form $-\infty . 0$

("An indeterminate product $0 . \infty$")

Idea: If we have $\lim_{x \to a} f(x) . g(x)$ where $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \pm \infty$, we can rewrite $\lim_{x \to a} f(x).g(x) = \lim_{x \to a} \frac{f(x)}{(1/g(x))} = \frac{0}{0}$ type

or $= \lim_{x \to a} \frac{g(x)}{(1/f(x))} = \frac{\infty}{\infty}$ type

and apply L'Hospital.

So back to Example: $\lim_{x \to -\infty} x.e^{x} = \lim_{x \to -\infty} \frac{x}{1/e^{x}}$

$\frac{\infty}{\infty}$, apply L'H. : $= \lim_{x \to -\infty} \frac{1}{-e^{-x}} = \frac{1}{\infty} = 0.$

OR $\lim_{x \to -\infty} x.e^{x} = \lim_{x \to -\infty} \frac{e^{x}}{(1/x)} \underset{\text{L'H. } \frac{0}{0}}{=} \lim_{x \to -\infty} \frac{e^{x}}{-x^{-2}}$

$= \ldots$ keeps getting worse applying L'H.

So try one way, where the derivatives will, you hope, simplify matters, but if it doesn't work out, don't be afraid to stop & try the other way.

<u>Example</u> Find $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x}$.

<u>Solution</u> This has indeterminate form 1^{∞}

$1 = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)$

$\infty = \lim_{x \to \infty} x$.

Can also get indeterminate forms 0^{0} & ∞^{0}.

How do we deal with any of these types?

<u>Idea</u> Convert an "<u>indeterminate power</u>" into an

indeterminate product using $\underline{\underline{\ln}}$. (& recover original limit using e)

If $\lim\limits_{x\to a} f(x)^{g(x)}$ has form 0^0, ∞^0, 1^∞, then

$\downarrow$

rewrite limit as $\lim\limits_{x\to a}\left(e^{\ln\left(f(x)^{g(x)}\right)}\right)$

Aside 0^∞ is NOT an indeterminate form: if $\lim\limits_{x\to a} f(x)=0$ and $\lim\limits_{x\to a} g(x)=\pm\infty$ then $\lim\limits_{x\to a}\left(f(x)^{g(x)}\right)=\begin{cases}0\\ \infty\end{cases}$ dep on what $g(x)$ does

$\left(= e^{\underbrace{\lim\limits_{x\to a}(g(x)\ln f(x))}_{\pm\infty \quad -\infty}} \to e^{\mp\infty}=\begin{cases}0\\ \infty\end{cases}\right)$

$$\lim_{x\to a}\left(e^{g(x)\cdot\ln f(x)}\right)$$

$$= e^{\underbrace{\lim\limits_{x\to a}(g(x)\cdot\ln f(x))}}$$

Use L'H. to deal with this.

Back to Example: $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

$$= e^{\underbrace{\lim\limits_{x\to\infty} x\cdot\ln\left(1+\frac{1}{x}\right)}}$$

$$\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{1/x}$$

$$\underset{\frac{0}{0}\text{ L'H.}}{=}\lim_{x\to\infty}\frac{\left(-1/x^2\right)/\left(1+1/x\right)}{-1/x^2}$$

$$=\lim_{x\to\infty}\frac{1}{1+1/x}=1$$

$$= e^1 = e.$$

Exercise Find $\lim_{x \to 0} \left(\tan(2x)\right)^x$.

Example Find $\lim_{x \to 0} \left(\cot(x) - \frac{1}{x}\right)$.

T.B.C.