1ZA3 (SECTION C01) - ENGINEERING MATHEMATICS I

Lecture 18

Last time

L'HOSPITAL'S RULE

If $\lim_{x\to a} \frac{f(x)}{g(x)}$ has indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then it equals $\lim_{x\to a} \frac{f'(x)}{g'(x)}$. ← if this limit exists or is $\pm\infty$.

Gives a strategy for tackling indeterminate products $0 \cdot \infty$ and indeterminate powers : ∞^0, 0^0, 1^∞ (NOT 0^∞) $= 0$ check this using ideas from last time!

This means $\lim_{x\to a} \left(f(x)^{g(x)}\right)$ where $\lim_{x\to a} f(x) = 1$, $\lim_{x\to a} g(x) = \pm\infty$

Indeterminate differences

Example Find $\lim_{x\to 0}\left(\cot(x) - \frac{1}{x}\right)$.

Solution $\lim_{x\to 0} \cot(x)$ is infinite, same as $\lim_{x\to 0} \frac{1}{x}$

So we have indeterminate difference $\infty - \infty$

Idea Try to convert to a quotient or power or product & if needed apply L'Hospital.

Here in the Exercise :

$$\lim_{x\to 0}\left(\cot(x) - \frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{\cos(x)}{\sin(x)} - \frac{1}{x}\right)$$

$$= \lim_{x \to 0} \left(\frac{x\cos(x) - \sin(x)}{x\sin(x)} \right) \leftarrow \frac{0}{0}$$

$$\overset{L'H \frac{0}{0}}{=} \lim_{x \to 0} \left(\frac{-x\sin(x) + \cancel{\cos(x)} - \cancel{\cos(x)}}{\sin(x) + x\cos(x)} \right) \leftarrow \frac{0}{0}$$

$$\overset{L'H \frac{0}{0}}{=} \lim_{x \to 0} \left(\frac{\overbrace{-x\cos(x) - \sin(x)}^{0}}{\underbrace{\cos(x)}_{1} - \underbrace{x\sin(x)}_{0} + \underbrace{\cos(x)}_{1}} \right)$$

$$= \frac{0}{2} = 0 .$$

4.5 Curve Sketching

Checklist (A) Domain – where is $f(x)$ defined?

(B) Intercepts – zeros ($f(x) = 0$) and $f(0)$

(C) Symmetry – $f(-x) = \begin{cases} -f(x) & (f \text{ odd}) \\ f(x) & (f \text{ even}) \end{cases}$

– $f(x + \underset{\text{period}}{T}) = f(x)$ – f periodic

How to find T? Suggested by the form of the function e.g. if there's a $\cos(3x)$, maybe $T = \frac{2\pi}{3}$?

(D) Asymptotes – behaviour of f at $\pm\infty$

(do we have $\lim_{x \to \pm\infty} f(x) = \#$ or $\pm\infty$) ← HA

VA: – are there points a for which $\lim_{x \to a} f(x) = \pm\infty$

(E) Increasing / Decreasing — where is $f'(x) = 0$?
& then $\gtrless 0$?

(F) Local Max. / Min. — where is $f'(x)$ undefined?

→ use 1st / 2nd derivative test.

(G) Concavity — where is $f''(x) = 0$ or undefined?
— where is $f''(x) \gtrless 0$? Where does concavity change (points of inflection)?

<u>Example</u> Sketch the graph of $f(x) = \dfrac{2x}{x^2+3}$.

<u>Solution</u> (A) Domain : $x \in \mathbb{R}$, $x \in (-\infty, \infty)$
all real #s x

(B) Intercepts : $f(x) = 0$ when $x = 0$
$f(0) = 0$
} $(0,0)$ is the only intercept

(C) Symmetry : $f(-x) = \dfrac{2(-x)}{(-x)^2+3} = \dfrac{-2x}{x^2+3} = -f(x)$

So $f(x)$ is an odd function

Nothing about $\dfrac{2x}{x^2+3}$ suggests periodicity — no trig. functions, say.

(D) Asymptotes : No vertical asymptotes (nowhere the function blows up).

$$\lim_{x\to\infty} \frac{2x}{x^2+3} \underset{\uparrow \atop \text{L'H } \frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{\cancel{2}1}{\cancel{2}x} = 0$$

EITHER check $\lim_{x \to -\infty} \frac{2x}{x^2+3} = 0$ OR use $f(x)$ odd function to determine this.

in the same way as above

So we have $y=0$ HA at $\pm\infty$.

(E) Increasing / Decreasing: Solve $f'(x) = 0$.

$$0 = f'(x) = \frac{2(x^2+3) - (2x)(2x)}{(x^2+3)^2} = \frac{-2x^2+6}{(x^2+3)^2}$$

$$= \frac{-2(x^2-3)}{(x^2+3)^2}$$

So $f'(x) = 0$ when $x^2 - 3 = 0$

i.e. $x = \pm\sqrt{3}$

Check regions $(-\infty, -\sqrt{3})$, $(-\sqrt{3}, \sqrt{3})$, $(\sqrt{3}, \infty)$:

(Notice $(x^2+3)^2 > 0$) so:

on $(-\infty, -\sqrt{3})$ $f'(x) < 0$ so f decreasing

on $(-\sqrt{3}, \sqrt{3})$ $f'(x) > 0$ so f increasing

on $(\sqrt{3}, \infty)$ $f'(x) < 0$ so f decreasing

(F) Local Max. / Min.

At $x = -\sqrt{3}$ $f'(x)$ goes from $-$ve to $+$ve $\rightarrow$ local min.

At $x = \sqrt{3}$ $f'(x)$ goes from $+$ve to $-$ve $\rightarrow$ local max.

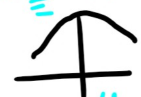
(9) Concavity   Solve $f''(x) = 0$

$$f''(x) = \dots = \frac{4x(x^2-9)}{(x^2+3)^3}$$

Notice $\} > 0$ as $x^2+3 > 0$ for all x

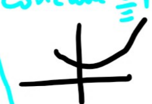
So $f''(x) = 0$ when $x = 0$, $x = \pm 3$

Concave Down ward



On $(-\infty, -3)$, $f''(x) < 0$, CDn

Concave Upward



on $(-3, 0)$, $f''(x) > 0$, CU

on $(0, 3)$, $f''(x) < 0$, CDn

on $(3, \infty)$, $f''(x) > 0$, CU

All 3 points $-3, 0, 3$ are points of inflection (concavity changes)

Finally we sketch the graph