1ZA3 (SECTION C01) - ENGINEERING MATHEMATICS I

Lecture 20

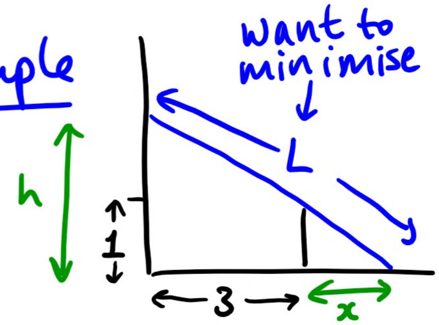
Last time — OPTIMIZATION PROBLEMS

→ Turning word problems into equations

→ Solution comes from maximising or minimising some $f(x)$

↳ finding absolute max./min.

Example



From $h^2+(x+3)^2=L^2$ and $\frac{x+3}{h}=\frac{x}{1}$ we saw

$$L^2=(x+3)^2\left(\frac{1}{x^2}+1\right)$$

a function of 1 variable

We could write $L=\sqrt{(x+3)^2\left(\frac{1}{x^2}+1\right)}$ & find abs. min.

or make our lives a little easier & find abs. min. of L^2 (& square root at end).

Look at domain of variables: $L>0$, $\underbrace{x>0}_{x\in(0,\infty)}$, $h>0$

Check critical #s of f^n $\underbrace{(x+3)^2\left(\frac{1}{x^2}+1\right)}_{=L^2}$ (no end points)

Find

$$(L^2)'=2(x+3)\left(\frac{1}{x^2}+1\right)+(x+3)^2\left(-\frac{2}{x^3}\right)=\ldots=\frac{2(x+3)(x^3-3)}{x^3}$$

$f'(x)$ is defined on $(0,\infty)$

$f'(x) = 0$ when $x^3 - 3 = 0$ $\rightarrow$ when $x = \sqrt[3]{3}$
(on $x > 0$)

1st deriv. test: on $(0, \sqrt[3]{3})$ $f'(x) < 0$
on $(\sqrt[3]{3}, \infty)$ $f'(x) > 0$ $\leftarrow$ local min.
(& no other critical # means L^2 greater everywhere else)

So L^2 minimised at $x = \sqrt[3]{3}$ $\rightarrow$ $L^2 \doteq 33.4$

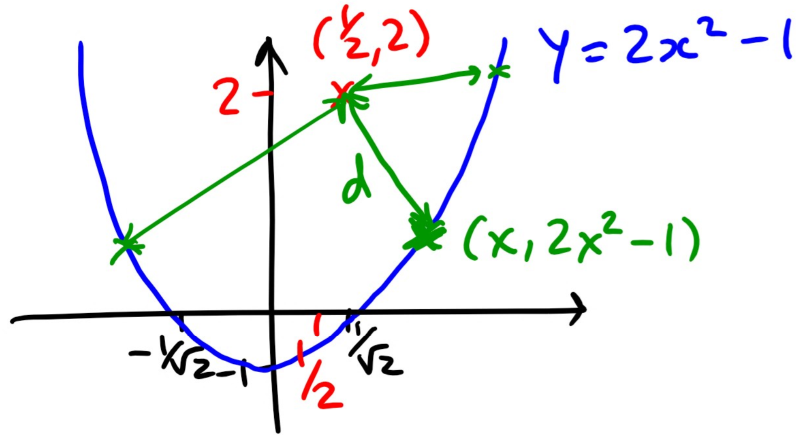
i.e. L minimised at $\underline{\underline{L \doteq 5.78\text{m}}}$

Example Find the point on the parabola $y = 2x^2 - 1$ that is closest to the point $(\frac{1}{2}, 2)$.

Solution Diagram:

distance $(a_1, b_1) \rightarrow (a_2, b_2)$

$\sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2}$



$$d = \sqrt{(x - \tfrac{1}{2})^2 + (2x^2 - 1 - 2)^2} = \sqrt{4x^4 - 11x^2 - x + 9\tfrac{1}{4}}$$

$\uparrow$ correct

Exercise

$= \ldots = \cancel{\sqrt{4x^4 - 15x^2 + 15\frac{1}{4}}}$ FAIL: Sorry :($\cancel{\left(= (4x^4 - 15x^2 + 15\frac{1}{4})^{1/2} \right)}$

(just made a mistake when simplifying – happens to us all...)

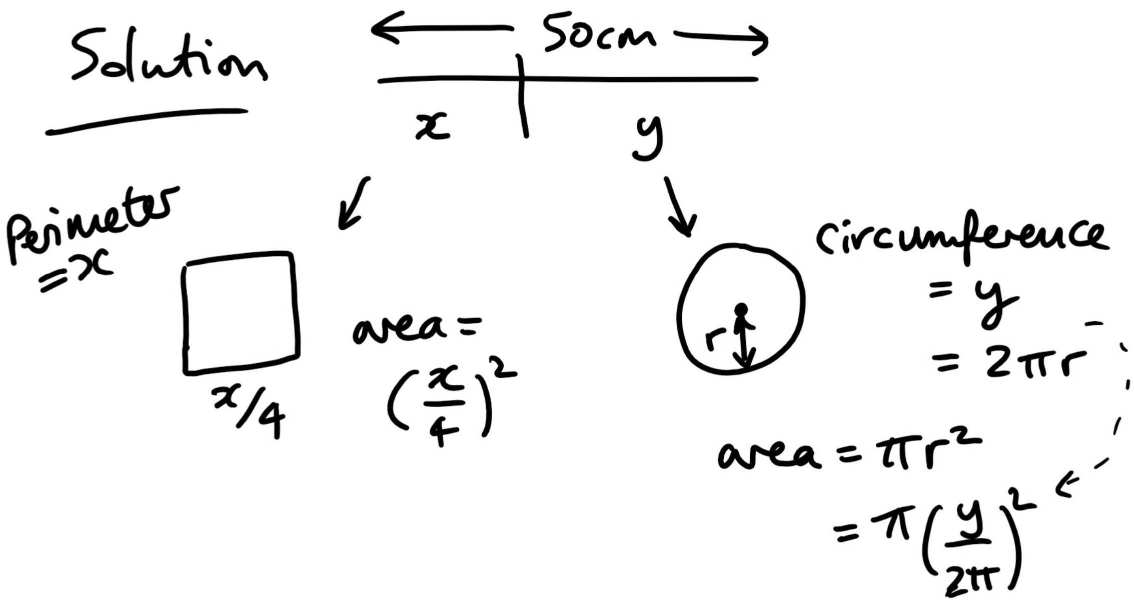
$\downarrow$ ... proceed exactly as in ladder problem from here

$\rightarrow$ minimise d (or d^2, if you prefer & remember to $\sqrt{\ }$ at end)

See separate document for solution/explanation.

Example A wire of 50 cm is cut into 2 pieces, with one piece made into a square and the other into a circle. Where should the wire be cut to maximise the TOTAL area of the 2 shapes?

Solution



$$50 = x + y$$

Total Area: $A = \left(\frac{x}{4}\right)^2 + \pi\left(\frac{y}{2\pi}\right)^2 = \frac{x^2}{16} + \frac{y^2}{4\pi}$

Now eliminate y: $y = 50 - x \longrightarrow$

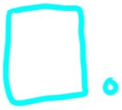
$$A = \frac{x^2}{16} + \frac{(50-x)^2}{4\pi}$$

What is $A' = \frac{x}{8} - \frac{(50-x)}{2\pi}$

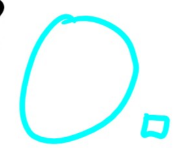
When is $A' = 0$? $\quad \frac{x}{8} - \frac{50-x}{2\pi} = 0$

$$\ldots \quad x = \frac{200}{4+\pi} \approx 28 \text{ cm}$$

That would give $A \approx 87.5 \text{ cm}^2$

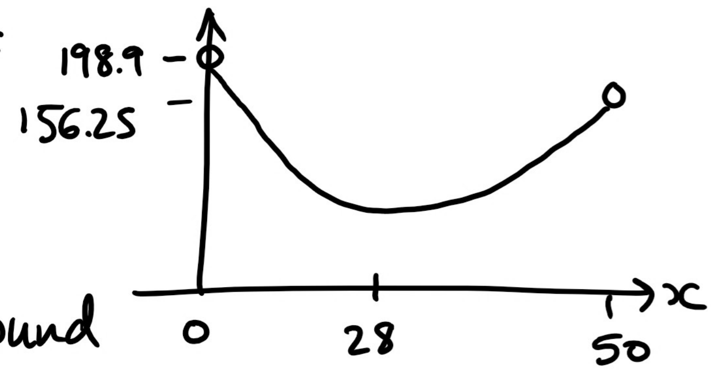
? If we cut wire close to end, made a big  square ($x \approx 50$) & tiny circle ($y \approx 0$)

What do we get? $A \cong \left(\frac{50}{4}\right)^2 = 156.25 \text{ cm}^2$

If we make a big circle & tiny square? 

Then $A \cong \pi\left(\frac{50}{2\pi}\right)^2 \approx 198.9 \text{ cm}^2$

Look at graph of A :



WARNING!!!!!

Be careful!

Even though we only found one critical value it was a minimum NOT max.

In fact <u>bad question</u> : there is NO maximum unless, say, we can use all the wire for the circle. Always check carefully & make sure.

4.9 Antiderivatives

Undoing — as much as we can — the act of differentiating a function.

<u>Definition</u> Given a function $f(x)$ on an interval I, an <u>antiderivative</u> $F(x)$ is any

function with $F'(x) = f(x)$ for all $x \in I$.

$\frac{dF}{dx}$

We write $F(x) = \int f(x)\,dx$.

← this is called an "integral sign".

Another name for antiderivative is <u>indefinite integral</u>

(We'll see later on what a "definite integral" is.)