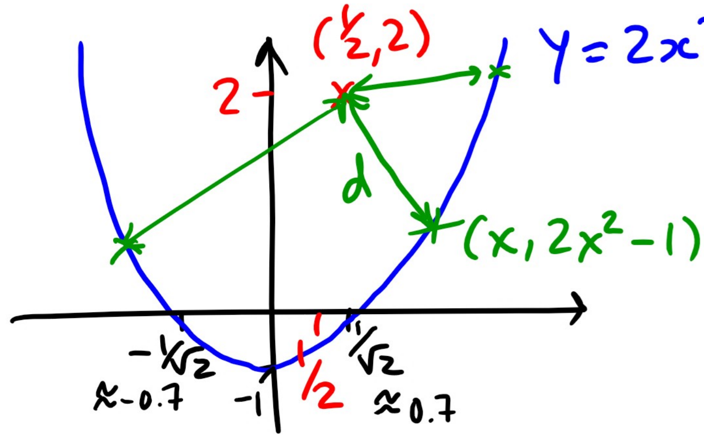
1ZA3 (SECTION C01) – ENGINEERING MATHEMATICS I

Exercise from Lecture 20

Exercise Find the point on the parabola $y = 2x^2 - 1$ that is closest to the point $(\frac{1}{2}, 2)$.

Solution

Diagram:



$$d = \sqrt{(x-\tfrac{1}{2})^2 + (2x^2-1-2)^2}$$

$$= \sqrt{(x-\tfrac{1}{2})^2 + (2x^2-3)^2}$$

$$= \sqrt{x^2 - x + \tfrac{1}{4} + 4x^4 - 12x^2 + 9}$$

$$= \sqrt{4x^4 - 11x^2 - x + 9\tfrac{1}{4}}$$

In order to minimise d, it is enough if we find the minimum of d^2 and then take the square root:

Minimise $(d(x))^2 = 4x^4 - 11x^2 - x + 9\frac{1}{4}$.

First solve for $\frac{d}{dx}(d(x)^2) = 0$. Differentiating, we have: $16x^3 - 22x - 1 = 0$

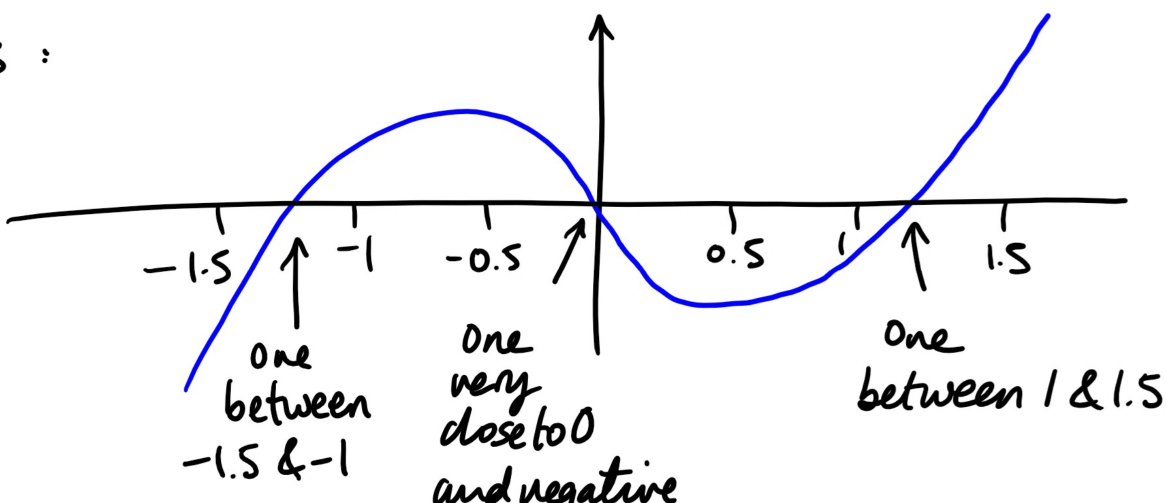
How are we going to find a solution to

$$16x^3 - 22x - 1 = 0 \ ?$$

In such situations, we might get lucky & be able to read off the answer. But otherwise (as here), what is a technique that you know for finding approximate solutions to such equations with great accuracy?

... Newton's Method.

A plot of this function tells you that there are 3 roots:



We're interested in finding the absolute minimum of d^2, so we're interested in finding a root of the derivative at which the derivative is increasing (i.e. going from negative to positive: graph shape ↘↗)

So the zero near $x=0$ is not interesting for us, but the other two zeros are.

So we run Newton's Method twice, once with $x_1 = -1$, say; the other time with $x_1 = 1$, say.

When we do that, we can find approximate solutions to $(d^2)' = 0$ of $x = -1.15$ and $x = 1.19$.

Plugging these into $d^2 = 4x^4 - 11x^2 - x + 9\frac{1}{4}$, we see that at $x \cong -1.15$, $d^2 \cong 2.848\ldots$, and at $x \cong 1.19$, $d^2 \cong 0.504\ldots$.

So we see that at $x \cong -1.15$ we must only have a local minimum, while at $x \cong 1.19$ we have not only a local min. but also an absolute minimum of d^2, and hence of d.

So the minimum value of d (at $x \cong 1.19$) is:

$$d \cong \sqrt{0.504} \cong \underline{\underline{0.71}}.$$

(Look back at the original diagram to see where the values of $x \cong -1.15$ and $x \cong 1.19$ lie & to see what is happening with d there.)