1ZA3 (SECTION C01) - ENGINEERING MATHEMATICS I

Lecture 21

Last time ANTIDERIVATIVES

Given $f(x)$ (defined on $x \in I$), an antiderivative of $f(x)$ is any function $F(x)$ with $F'(x) = f(x)$.
(for all x in I)

Another name for antiderivative: indefinite integral

Notation: $F(x) = \int f(x)\,dx$.

Exercise Write down an antiderivative of x^5.

Solution $\frac{x^6}{6}$, $\frac{x^6}{6} + 32$, $\frac{x^6}{6} + 5700$

say $\frac{x^6}{6} + C$

Suppose we have $f(x)$ with two antiderivatives $F(x)$, $G(x)$, then

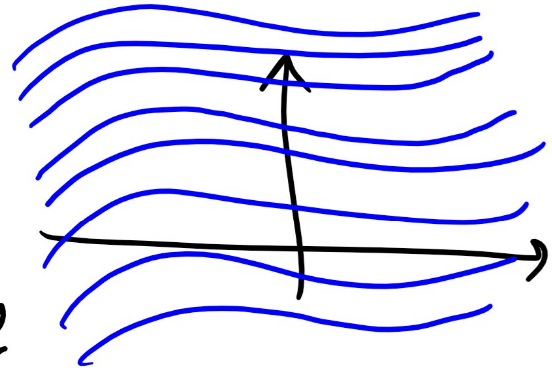
$$\frac{d}{dx}(G(x) - F(x)) = \frac{dG}{dx} - \frac{dF}{dx} = f(x) - f(x) = 0$$

So (by same reasoning using MVT as with inc./dec. test)

$G(x) - F(x)$ is constant i.e. $G(x) - F(x) = C$ (some constant)

$$G(x) = F(x) + C$$

So in general all antiderivatives of a function $f(x)$ are vertical translates of one another (all differ by constant amounts)



So for our example of $f(x) = x^5$, the "family" of antiderivatives is $\frac{x^6}{6} + C$ (nothing else possible).

Just as with differentiation, we'll have rules to build antiderivatives of complicated functions from constituent parts.

Rules (beginning) → In the next weeks, we'll look at rules for "undoing" all those differentiation rules – product rule, quotient rule, even chain rule...

For example if $F(x) = \int f(x)\,dx$ and $G(x) = \int g(x)\,dx$

then $F(x) \pm G(x) = \int (f(x) \pm g(x))\,dx$

$$= \int f(x)\,dx \pm \int g(x)\,dx$$

Also $\int k f(x)\,dx = k \int f(x)\,dx$. $\left((kF(x))' = kF'(x)\right)$

k constant

(See tables on p. 352, p. 403 of textbook.)

Function	Antiderivative
x^n	$\dfrac{x^{n+1}}{n+1} + C$
k	$kx + C$
$\cos(x)$	$\sin(x) + C$
$\sin(x)$	$-\cos(x) + C$
$\sec^2(x)$	$\tan(x) + C$
$\sec(x)\tan(x)$	$\sec(x) + C$
e^x	$e^x + C$
b^x	$\dfrac{1}{(\ln b)} b^x + C$
$\dfrac{1}{1+x^2}$	$\arctan(x) + C$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin(x) + C$
$\sinh(x)$	$\cosh(x) + C$
$\cosh(x)$	$\sinh(x) + C$
$\dfrac{1}{x}$	$\ln\lvert x\rvert + C$

$\int \frac{1}{x}\,dx = \ln x + C$ for $x > 0$

What if $x < 0$? What is $\frac{d}{dx}(\ln(-x))$?

$$= \frac{-1}{-x} = \frac{1}{x}$$

So $\int \frac{1}{x}\, dx = \ln(-x) + C$ for $x < 0$

But what is $= \begin{cases} x & \text{for } x > 0 \\ -x & \text{for } x < 0 \end{cases}$? $= |x|$

Example Find all antiderivatives of

$$f(x) = \frac{6x^7 + \sqrt[3]{x}}{x^2} - 2\sec^2(x).$$

Solution Simplify: $f(x) = 6x^5 + x^{1/3-2} - 2\sec^2(x)$

$$= 6x^5 + x^{-5/3} - 2\sec^2(x)$$

So $\int f(x)\,dx = \int (6x^5 + x^{-5/3} - 2\sec^2(x))\,dx$

$$= \int 6x^5 dx + \int x^{-5/3} dx - \int 2\sec^2(x) dx$$

$$= 6\int x^5 dx + \int x^{-5/3} dx - 2\int \sec^2(x) dx$$

$$= 6.\left(\frac{x^6}{6}\right) + C + \frac{x^{-5/3+1}}{(-5/3+1)} + C - 2\tan(x) + C + C$$

$$= x^6 - \left(\frac{3}{2}\right) x^{-2/3} - 2\tan(x) + C$$

Example (Initial Value Problem) Throw a ball into the air with an initial speed of 5 m/s. Find position and velocity as a function of time t.

Solution

(up is positive)

$v(t) = 5$ at $t = 0$

$a(t) = -10 = -g$

If $v'(t) = a(t)$ then $v(t) = \int a(t)\,dt$

$$= \int -10\,dt$$

$$= -10t + C$$

Solve for C: $v(0) = 5 = \cancel{-10(0)} + C$

$\cancel{-10(0)} \to 0$

$$\Rightarrow C = 5$$

$$\underline{v(t) = -10t + 5}$$

i.e. the initial velocity tells us which of the possible antiderivatives for $a(t)$ we really have

$p'(t) = v(t)$

So $p(t) = \int v(t)\,dt = \int -10t + 5\,dt$

$$= -10\int t\,dt + 5\int 1\,dt$$

$$= -10\frac{t^2}{2} + 5t + C$$

$$= -5t^2 + 5t + C$$

← To find C, we need an initial position.

So now assume initial position is 1m high.

i.e. $p(0) = 1$, so solve for C:

$$p(0) = 1 = \cancel{-5(0)^2} + \cancel{5(0)} + C$$

$\cancel{-5(0)^2} \to 0$, $\cancel{5(0)} \to 0$

$\Rightarrow C = 1$ so $p(t) = -5t^2 + 5t + 1$.

Now Part B When does the ball hit the ground?

Solution When $p(t) = 0$.

Solve for t in $-5t^2 + 5t + 1 = 0$

A computational error, it turns out — sorry! Fortunately you all know how to solve the quadratic formula!!

Quadratic: $t \simeq \cancel{\frac{-5}{2} \pm \frac{11.38}{2}}$ $\frac{-5}{-10} \pm \frac{\sqrt{45}}{10}$

$t > 0$ So only +ve answer possible

So $t \simeq \cancel{3.09\,\text{s}}$. $1.17\,\text{s}$