1ZA3 (SECTION C01) - ENGINEERING MATHEMATICS I

Lecture 22

Today Sigma Notation : Writing Sums

→ compact & convenient way of writing sums

Example $1+4+9+16+25+36$

$= 1^2+2^2+3^2+4^2+5^2+6^2$

$= \sum_{i=1}^{6} i^2$ ← "The sum of i^2 from i equals 1 to 6" or "The sum from i equals 1 to 6 of i^2."

i is the "index"

and it starts at bottom # & goes up 1 each term & stops at top #.

Example $\sum_{i=1}^{3} \frac{1}{i} = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}$

Example $\sum_{k=7}^{11} 2^k = 2^7+2^8+2^9+2^{10}+2^{11}$

Example $\sum_{k=0}^{5} 3 = 3+3+3+3+3+3$

(0 1 2 3 4 5) ← k

Example $\sum_{i=1}^{n} (i-1) = (1-1)+(2-1)+(3-1)+\dots+((n-1)-1)+(n-1)$

<u>Warning</u> : don't get mixed up between the index letter & any other unknown parameters denoted by letters.

Go back to last Example :

$$\sum_{i=1}^{n} i-1 = (1-1) + (2-1) + \ldots + (n-1)$$

$$= 0 + 1 + 2 + \ldots + (n-1)$$

$$= \sum_{i=0}^{n-1} i$$

(Also happens to $= \sum_{i=1}^{n-1} i$ here as first term $= 0$)

"Index-Shifting" : there's no one unique way to write any sum

<u>Example</u>

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2$$

$$= \sum_{i=4}^{n+3} (i-3)^2 = (4-3)^2 + (5-3)^2 + (6-3)^2 + \ldots +$$

→ up by 3

↓ down by 3

↘ up by 3

"What goes up, must come down"

$$(\underbrace{?}_{n}-3)^2$$

$$= \sum_{i=-1}^{n-2} (i+2)^2 = (-1+2)^2 + (0+2)^2 + \ldots + (n-2+2)^2$$

$$= 1^2 + 2^2 + \ldots + n^2$$

<u>Example</u>

$$\sum_{i=1}^{5} 3i = 3.1 + 3.2 + 3.3 + 3.4 + 3.5$$

$$= 3(1+2+3+4+5)$$

$$= 3\sum_{i=1}^{5} i$$

We can pull anything (any factor) that does not depend on the index parameter through the Σ.

In general we refer to terms in a sum as expressions like a_i or b_i (like a function of i but takes integer inputs)

So above can be written: for any constant (relative to i)

$$\sum_{i=m}^{n} c a_i = c \sum_{i=m}^{n} a_i$$

Likewise

$$\sum_{i=m}^{n} a_i \pm \sum_{i=m}^{n} b_i = (a_m + a_{m+1} + a_{m+2} + \dots + a_n) \pm (b_m + b_{m+1} + b_{m+2} + \dots + b_n)$$

$$= (a_m \pm b_m) + (a_{m+1} \pm b_{m+1}) + \dots + (a_n \pm b_n)$$

$$= \sum_{i=m}^{n} (a_i \pm b_i)$$

These give us rules to understand complicated sums in terms of simpler pieces.

Formulas (a)

$$\sum_{i=m}^{n} 1 = \overset{m}{1} + \overset{m+1}{1} + \dots + \overset{n}{1} = n - (m-1)$$

Please see end of file with worked solutions to exercises for more about this.

$$\sum_{i=1}^{n} 1 = \overset{1}{1} + \overset{2}{1} + \dots + \overset{n}{1} = n$$

There are n ones.

(b) $\sum_{i=1}^{n} i = 1+2+3+\ldots+n = \dfrac{n(n+1)}{2}$

$$\begin{array}{l} \quad 1+2+3+\ldots+n \\ + \; n+(n-1)+(n-2)+\ldots+1 \end{array}$$

add up column by column

We write the sum twice & just reverse the order the Second time

$2 \times \text{sum} = (n+1)+(n+1)+(n+1)+\ldots+(n+1) = n.(n+1) \rightarrow$ So sum is 1× half that amount

(c) $\sum_{i=1}^{n} i^2 = 1^2+2^2+3^2+\ldots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(d) $\sum_{i=1}^{n} i^3 = 1^3+2^3+3^3+\ldots+n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$

Exercise What is $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{5}{n}\left(\left(\frac{i+1}{n}\right)^2 + \frac{2i}{n}\right)$?

→ Focus on sum

→ Break it up into smaller manageable pieces

→ Remember i is the index letter.

Telescoping Sums

"collapse"

Example Find $\sum_{i=1}^{53}\left(\frac{1}{i} - \frac{1}{i+1}\right)$.

Solution $\sum_{i=1}^{53}\left(\frac{1}{i} - \frac{1}{i+1}\right) = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right)$

$\cancel{\ldots} + \left(\cancel{\frac{1}{52}} - \cancel{\frac{1}{53}}\right) + \left(\cancel{\frac{1}{53}} - \frac{1}{54}\right)$

$= 1 - \frac{1}{54} = \underline{\underline{53/54}}.$

Exercise Find $\sum_{i=1}^{10}\left(i^2-(i+2)^2\right)$.