1ZA3 (SECTION C01) - ENGINEERING MATHEMATICS I

Lecture 23

Last time Sigma Notation

$$\sum_{i=m}^{n} a_i = \text{"the sum from } i=m \text{ to } n \text{ of } a_i\text{"}$$

Like a function of i where i takes integer inputs

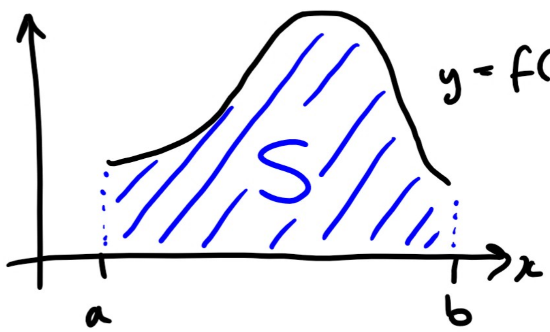
Useful:

$$\sum_{i=1}^{n} 1 = n \; ; \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \; ; \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \; ; \quad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

5.1 Areas (and Distances)

$\to v(t).t$

Area Problem



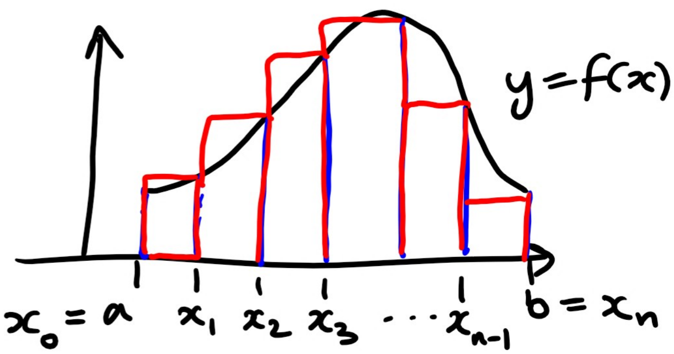
Given a positive continuous function $f(x)$, find the area under the curve $y=f(x)$ between $x=a$ and $x=b$.

This means "above the x-axis"

S = region under curve $y=f(x)$

Goal Find A = area of S.

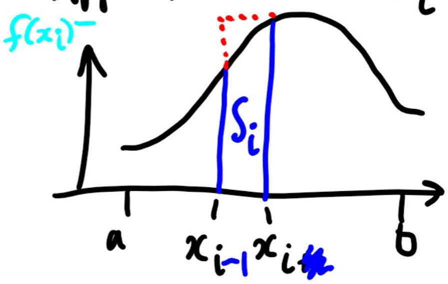
Big Idea Cut up S into thin vertical slices and approximate each slice by a rectangle.



Sum of areas of rectangles $\approx A$

Divide up $[a, b]$ into n mini-intervals each of which has same width $\Delta x = \dfrac{b-a}{n}$

Approximate S_i (ith slice) = slice under $y=f(x)$ between $x = x_{i-1}$ and $x = x_i$



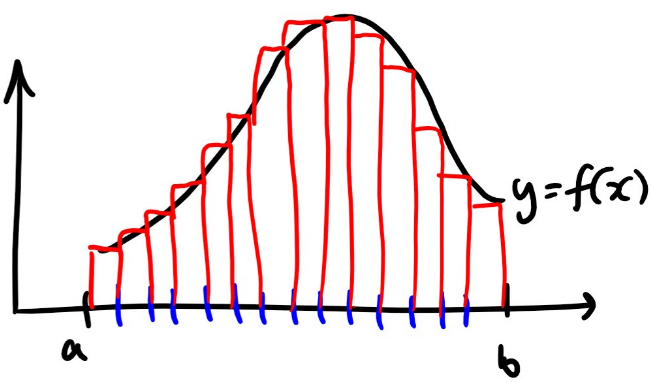
with rectange of width Δx and height $f(x_i)$.

Then

$$A \cong R_n = \Delta x . f(x_1) + \Delta x . f(x_2) + \ldots + \Delta x . f(x_n)$$

right endpoints of subintervals

$$= \sum_{i=1}^{n} \Delta x . f(x_i).$$



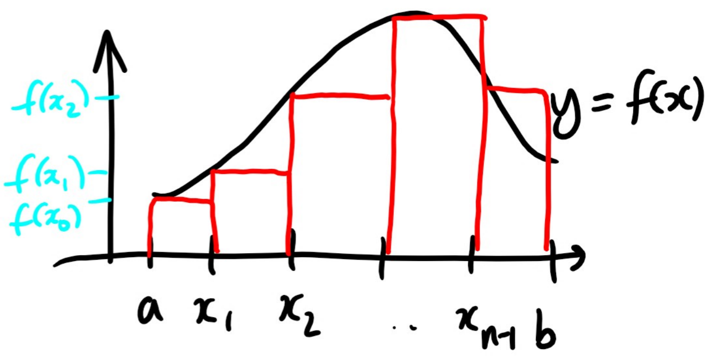
As we increase n (i.e. the # of slices) we get a better and better approximation to A.

We take this to the limit!!

We <u>define</u> the area A under $y=f(x)$ from $x=a$ to $x=b$ to be $\lim\limits_{n\to\infty} \sum_{i=1}^{n} \Delta x\, f(x_i)$.

[<u>Note</u> It can be shown that this limit exists since $f(x)$ is continuous.]

No good reason to approximate S_i with a rectangle whose height was $f(x_i)$ and not $f(x)$ for some other $x \in [x_{i-1}, x_i]$ e.g. could choose left endpoint x_{i-1}:



Area of $S_i \cong \Delta x\, f(x_{i-1})$

So $A \cong L_n = \sum_{i=1}^{n} \Delta x\, f(x_{i-1})$

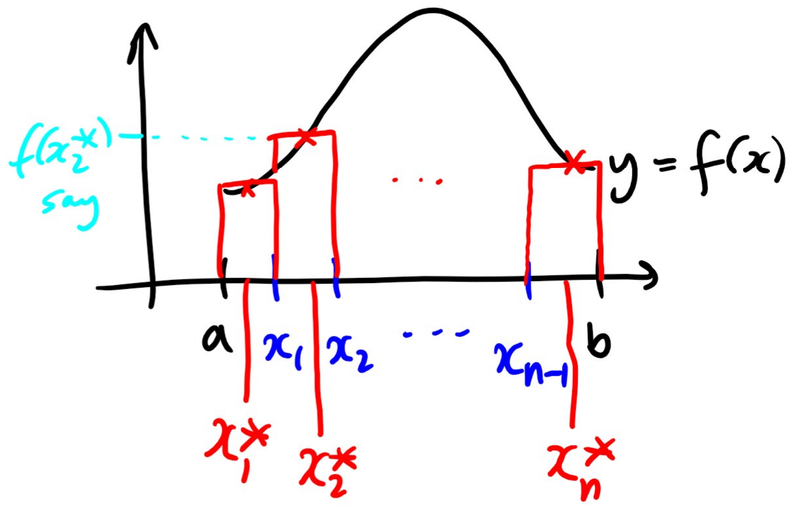
left endpoints

Then since $f(x)$ is continuous, we have

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x\, f(x_{i-1}).$$

Or we could take <u>any</u> so-called <u>sample point</u> in $[x_{i-1}, x_i]$ called x_i^* and use rectangle height $f(x_i^*)$ (and width Δx) to approximate S_i

e.g. x_i^* midpoint of $[x_{i-1}, x_i]$:



Then area of S_i

$$\cong f(x_i^*) . \Delta x$$

and $$A \cong \underbrace{\sum_{i=1}^{n} f(x_i^*)\Delta x}_{\text{Riemann Sum}}$$

and $$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x .$$

(This is all valid here as $f(x)$ continuous.)

<u>Example</u> Let $f(x) = (x+2)^3$.

Estimate A the area under $y = f(x)$ on the interval $[-1, 2]$ using $\underset{\overset{\shortparallel}{n}}{\text{six}}$ rectangles and $\underset{\text{sample points}}{\underset{\downarrow}{\text{midpoints}}}$.

<u>Solution</u>

First divide $[\overset{a}{\overset{\shortparallel}{-1}}, \overset{b}{\overset{\shortparallel}{2}}]$ into $\overset{n}{\overset{\shortparallel}{6}}$ subintervals of equal width $\Delta x = \dfrac{b-a}{n} = \dfrac{2-(-1)}{6} = \dfrac{1}{2}$.

So intervals are $[-1, -\frac{1}{2}], [-\frac{1}{2}, 0], [0, \frac{1}{2}], [\frac{1}{2}, 1], [1, \frac{3}{2}],$

$[\frac{3}{2}, 2]$.

T.B.C.