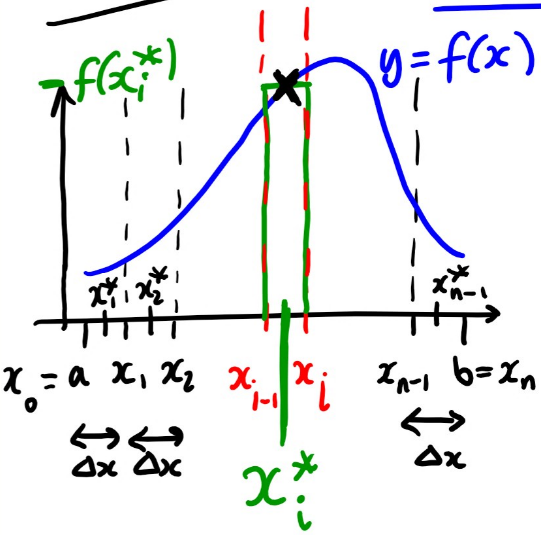
1ZA3 (SECTION C01) - ENGINEERING MATHEMATICS I

Last time

The Area Problem & Riemann Sums



A = area under $y=f(x)$

$$= \lim_{n\to\infty} \boxed{\sum_{i=1}^{n} \Delta x\, f(x_i^*)}$$

where x_i^* is a sample point in $[x_{i-1}, x_i]$

and $\Delta x = \dfrac{b-a}{n}$.

$$\underbrace{\Delta x}_{\frac{b-a}{n}}\ \underbrace{f(x_i^*)}$$

Also very much depends on n.

Where the x_i^*'s can be placed, & hence the value of $f(x_i^*)$, depends on how many subintervals there are i.e. the # n.

$x_i^* \in [x_{i-1}, x_i]$

$x_0 = a$

$x_1 = a + \dfrac{b-a}{n}$

$x_2 = a + \dfrac{b-a}{n} + \dfrac{b-a}{n} = a + \dfrac{2(b-a)}{n}$

$x_3 = a + \dfrac{2(b-a)}{n} + \dfrac{b-a}{n} = a + \dfrac{3(b-a)}{n}$

$\vdots$

$$\boxed{x_i = a + \frac{i(b-a)}{n}}$$

Useful formula to remember!

Back to Example $f(x) = (x+2)^3$

Estimate area A under $y=f(x)$ on $[-1,2]$ using 6 rectangles and midpoints.

Solution Rectangles : $[-1,-\frac{1}{2}], [-\frac{1}{2},0], [0,\frac{1}{2}], [\frac{1}{2},1], [1,\frac{3}{2}], [\frac{3}{2},2]$

$\boxed{\Delta x = 1/2}$

Sample points: Midpoints : $-\frac{3}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}$

$x_1^*, \quad x_2^*, \quad x_3^*, \quad x_4^*, \quad x_5^*, \quad x_6^*$

Heights of corresponding rectangles: $f(-\frac{3}{4}), f(-\frac{1}{4}), f(\frac{1}{4}), f(\frac{3}{4}), f(\frac{5}{4}), f(\frac{7}{4})$

$= (\frac{5}{4})^3, (\frac{7}{4})^3, \ldots$ etc.

$$A \cong \sum_{i=1}^{6} \Delta x . f(x_i^*) = \frac{1}{2}\left(\frac{5}{4}\right)^3 + \frac{1}{2}\left(\frac{7}{4}\right)^3 + \frac{1}{2}\left(\frac{9}{4}\right)^3 + \frac{1}{2}\left(\frac{11}{4}\right)^3 + \frac{1}{2}\left(\frac{13}{4}\right)^3 + \frac{1}{2}\left(\frac{15}{4}\right)^3$$

$$= \underbrace{\frac{1}{2}\left(\frac{1}{4}\right)^3}_{\frac{1}{128}} \underbrace{(5^3 + 7^3 + 9^3 + 11^3 + 13^3 + 15^3)}_{=8100}$$

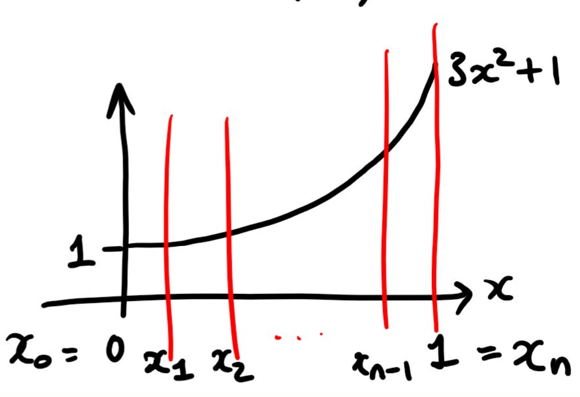
$$= \frac{8100}{128} = \frac{2025}{32} \cong \underline{63.28}.$$

Could pull out Δx; writing it as $\Delta x f(x_i^*)$ is to remind us we are summing up areas of rectangles.

Example Let $f(x) = 3x^2 + 1$ on $[0,1]$.

Find A, area under $y = f(x)$.

Solution



First find $\Delta x = \frac{b-a}{n}$

$= \frac{1-0}{n} = \frac{1}{n}$.

Use right endpoints:

$x_i^* = x_i = a + i\Delta x$ (from above)

So $x_i^* = \dfrac{i}{n}$

Height of ith rectangle is $f\left(\frac{i}{n}\right) = 3\left(\frac{i}{n}\right)^2 + 1$

So now $$A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \cdot f\left(\frac{i}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(3\left(\frac{i}{n}\right)^2 + 1\right).$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{3i^2}{n^2} + 1\right) = \frac{1}{n}\left(\sum_{i=1}^{n}\frac{3i^2}{n^2} + \sum_{i=1}^{n} 1\right)$$

$$= \frac{1}{n}\left(\frac{3}{n^2}\sum_{i=1}^{n} i^2 + n\right)$$

$$= \frac{1}{n}\left(\frac{3}{n^2}\cdot\frac{n(n+1)(2n+1)}{6} + n\right)$$

$$= \frac{1}{n}\left(\frac{\cancel{3}(2n^3 + 3n^2 + n)}{2\cancel{6}n^2} + n\right)$$

$$= \frac{2n^3 + 3n^2 + n}{2n^3} + 1$$

Now take limit: $$A = \lim_{n\to\infty}\left(\frac{2n^3 + 3n^2 + n}{2n^3} + 1\right)$$

$$= \lim_{n\to\infty}\left(\frac{2 + 3/n + 1/n^2}{2}\right) + 1 = 1 + 1 = \underline{\underline{2}}.$$

5.2 The Definite Integral

If $f(x)$ is defined on $[a,b]$, $\Delta x = \frac{b-a}{n}$, $[a,b]$ is divided into n subintervals of length Δx, and x_i^* is any sample point in ith subinterval $[x_{i-1}, x_i]$, then

the <u>definite integral</u> of $f(x)$ from $x=a$ to $x=b$

is $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x . f(x_i^*)$$

a #!!

as long as this limit exists & always gives same # regardless of choice of sample points.

[This happens if $f(x)$ is continuous or $f(x)$ has only finitely many jump discontinuities.]

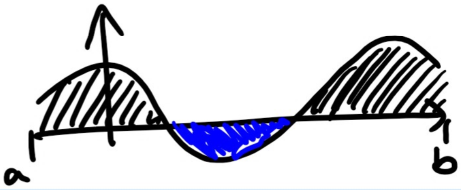
If this limit exists then we say $f(x)$ is <u>integrable on $[a,b]$</u>.

<u>Interpretation</u>

If $f(x)$ is integrable & positive, then

$\int_a^b f(x)\,dx$ is area under $y=f(x)$.
(& above x-axis).

If $f(x)$ is sometimes >0 & sometimes <0



$$\int_a^b f(x)\,dx = (\text{area under } y=f(x) \text{ above } x\text{-axis}) - (\text{area above } y=f(x) \text{ below } x\text{-axis})$$

$=$  $-$