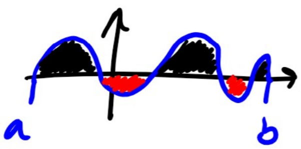
1ZA3 (SECTION C01) — ENGINEERING MATHEMATICS I

Lecture 25

Last time **The DEFINITE Integral**



$$\int_a^b f(x)\,dx = \text{(area under } y=f(x) \text{ above } x\text{-axis between } x=a \text{ and } x=b\text{)}$$
$$- \text{(area above } y=f(x) \text{ below } x\text{-axis between } x=a \text{ and } x=b\text{)}$$

$$\| $$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \Delta x\, f(x_i^*)$$

IF f(x) integrable

i.e. IF limit exists & is same for all choices of samplepoints x_i^*

e.g. f(x) continuous

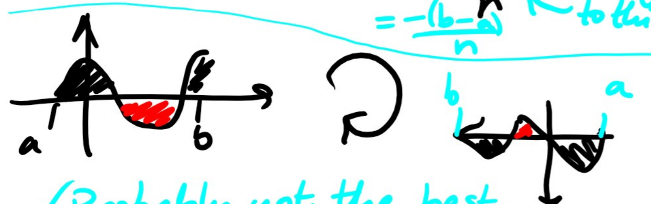
Properties of the definite integral

① $$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

common factor in terms of Riemann sum;

$$\Delta x = \frac{(b-a)}{n} \longrightarrow \frac{(a-b)}{n} = \frac{-(b-a)}{n}$$

if we swap a & b roles, we change to this



(Probably not the best way to think about it though!)

② $$\int_a^a f(x)dx = 0$$

③ $$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

Properties inherited from Σ notation rules & limit laws.

④ $$\int_a^b c f(x)dx = c\int_a^b f(x)dx$$ for a constant c not depending on x

⑤ $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

Area from a to c — Area from c to b — Area from a to b

⑥ If $f(x) \geq 0$ on $[a,b]$, then $\displaystyle\int_a^b f(x)\,dx \geq 0$

⑦ If $f(x) \geq g(x)$ on $[a,b]$, then $\displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

BIG REMARK: x is "dummy variable" in the notation

i.e. $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(s)\,ds = \int_a^b f(r)\,dr = \int_a^b f(y)\,dy$

These are always a <u>number</u> (and the same number!)

<u>Example</u> Use properties of the definite integral to find $\displaystyle\int_{-3}^{0} \sqrt{9-x^2} - 1\,dx$.

<u>Solution</u>

$$\int_{-3}^{0} \sqrt{9-x^2} - 1\,dx = \underbrace{\int_{-3}^{0} \sqrt{9-x^2}\,dx}_{\frac{9\pi}{4}} + \underbrace{\int_{-3}^{0} -1\,dx}_{-3}$$

Area $= \frac{1}{4} \cdot \pi \cdot 9$

(= $\frac{1}{4}$ circle area i.e. $\frac{1}{4}\pi r^2$)

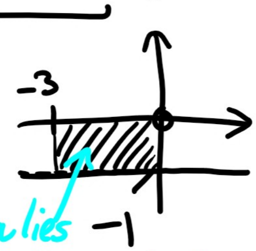
← Area lies above x-axis so is POSITIVE $= \frac{9\pi}{4}$.

$y = \sqrt{9-x^2}$

$y^2 = 9 - x^2$

$x^2 + y^2 = 9$

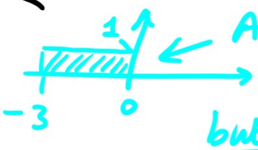
circle of radius 3, centre Origin



Area lies BELOW x-axis so is NEGATIVE $= -3$

$$= \frac{9\pi}{4} - 3.$$

Note $\displaystyle\int_{-3}^{0} -1\,dx \overset{④}{=} -\int_{-3}^{0} 1\,dx$ ← This is an OK way



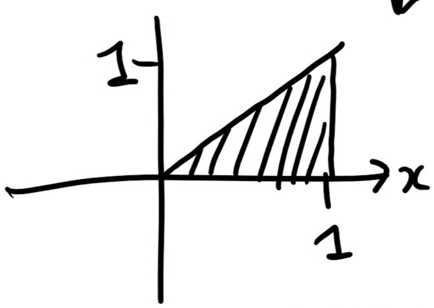
Area above so POSITIVE but subtracted

and $\displaystyle = -\int_{0}^{-3} -1\,dx = \int_{0}^{-3} 1\,dx$. ← But this is a formal exercise then & probably more confusing to think about.

Example Evaluate $\displaystyle\int_0^1 3x^2 + x + 1\,dx$.

Solution $\displaystyle\int_0^1 3x^2+x+1\,dx = \underbrace{\int_0^1 3x^2+1\,dx}_{=2 \text{ from earlier Riemann sum example}} + \underbrace{\int_0^1 x\,dx}$

As we already did the hard work to find this in class, I decided to let us use it.



Area $= \frac{1}{2}$

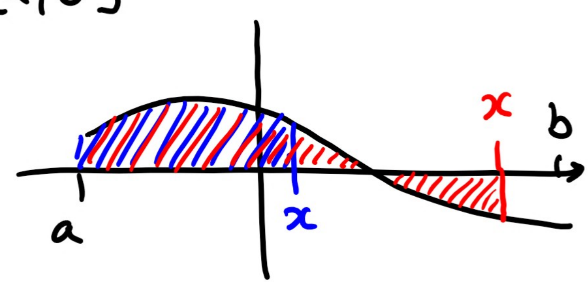
$$= 2 + \frac{1}{2} = \frac{5}{2}.$$

5.3 The Fundamental Theorem of Calculus

Let $f(t)$ be defined on $[a,b]$

Look at "area up to # x"

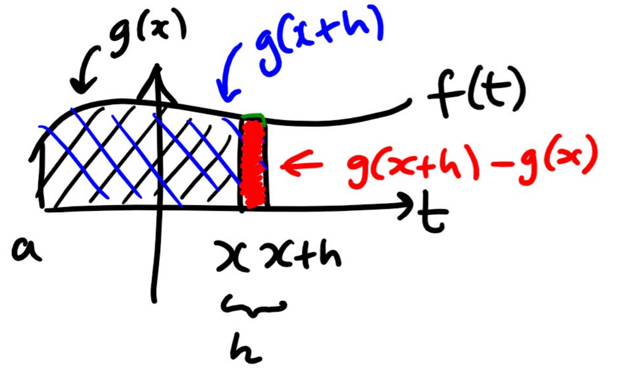
$$= \int_a^x f(t)\,dt$$



This is a function $g(x) = \int_a^x f(t)\,dt$. (It's value changes as x varies!)

What is the derivative of $g(x)$?

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \quad \text{(if the limit exists)}$$



$$= f(x)$$

(If limit exists.)

$$g(x+h) - g(x) \approx h \cdot f(x)$$

$$\frac{g(x+h) - g(x)}{h} \approx f(x)$$

Fundamental Theorem of Calculus Part I

If $f(x)$ is continuous on $[a,b]$, then $g(x) = \int_a^x f(t)\,dt$ is also continuous on $[a,b]$, differentiable on (a,b) and

same

not x (something different than variable up top)

$$g'(x) = f(x).$$

notice: x

Example Let $g(x) = \int_2^x t\sin(3t)\,dt$. Find $g'(x)$.

Solution $f(t) = t\sin(3t)$. By FTC $g'(x) = x\sin(3x)$.

i.e. we put in the right variable.

Example Let $h(x) = \int_5^{x^3} \frac{ds}{s^2}$. Find $\frac{dh}{dx}$.

x^3?

another way of writing $= \frac{1}{s^2}ds$

$= h'(x)$, of course.

Solution If $g(u) = \int_5^u \frac{ds}{s^2}$, then by FTC

$f(s) = \frac{1}{s^2}$

$g'(u) = f(u)$
$= \frac{1}{u^2}$.

But notice $h(x) = g(u(x))$ where $u(x) = x^3$

So $\frac{dh}{dx} = \frac{dg}{du} \cdot \frac{du}{dx} = \frac{1}{u^2} \cdot 3x^2 = \frac{1}{(x^3)^2} \cdot 3x^2 = \frac{3}{x^4}$.

Chain Rule!!

In general, if $h(x) = \int_a^{g(x)} f(t)\,dt$, then — by the Chain Rule —

$$h'(x) = f(g(x)) \cdot g'(x).$$

Think about this! Ask someone if you can't figure out where the formula comes from!

But there's more!!! $\int_a^x f(t)\,dt$

FTC Part I tells us $g(x)$ is an antiderivative of $f(x)$ $(g'(x) = f(x))$

Take any antiderivative $F(x)$ of $f(x)$.

We know F & g differ by a constant i.e.

$F(x) = g(x) + C$.

So now look at net change of F from a to b:

$$F(b) - F(a) = \left(g(b) + \cancel{C}\right) - \left(g(a) + \cancel{C}\right)$$

$$= g(b) - g(a) \quad \leftarrow \cancel{\int_a^a f(t)\,dt} \to 0$$

$$= \int_a^b f(t)\,dt.$$

<u>FTC Part II</u> If $f(x)$ is continuous on $[a,b]$ then

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

where $F(x)$ is <u>any</u> antiderivative of $f(x)$. Wow!