1ZA3 (SECTION C01) - ENGINEERING MATHEMATICS I

Lecture 26

Last time The FUNDAMENTAL THEOREM of Calculus

If $f(x)$ is continuous on $[a,b]$, then:

PART I: $g(x) = \int_a^x f(t)\,dt$ has $g'(x) = f(x)$;

PART II: $\int_a^b f(t)\,dt = F(b) - F(a)$, where $F(x)$ is any antiderivative of $f(x)$ (i.e. $F'(x) = f(x)$)

usually write this as $\Big[F(x)\Big]_a^b$ or $F(x)\Big]_a^b$ or $F(x)\Big|_a^b$

Example Find $\int_0^1 3x^2 + 1\,dx$. (See Riemann sum)

Solution $\int_0^1 3x^2+1\,dx = \Big[x^3 + x\Big]_0^1 = (1^3+1) - (0^3+0)$

$= 2.$

$F(x) = x^3 + x$ is AN antiderivative of $f(x) = 3x^2+1$.

Example Find $\int_5^{10} \frac{1}{x} - e^x\,dx$.

Solution $\displaystyle\int_5^{10} \frac{1}{x} - e^x \, dx = \int_5^{10} \frac{1}{x}\, dx - \int_5^{10} e^x \, dx$

$$= \Big[\ln|x|\Big]_5^{10} - \Big[e^x\Big]_5^{10}$$

$$= (\ln 10 - \ln 5) - (e^{10} - e^5)$$

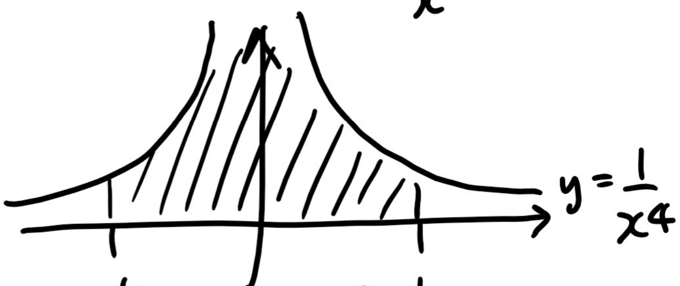
$$= \#$$

Example Find $\displaystyle\int_{-1}^{1} \frac{1}{x^4}\, dx$.

Solution $\displaystyle\int_{-1}^{1} \frac{1}{x^4}\, dx = \left[-\frac{x^{-3}}{3}\right]_{-1}^{1} = -\frac{1}{3(1)^3} - \left(-\frac{1}{3(-1)^3}\right)$

x^{-4} $\qquad -\dfrac{1}{3x^3}$

$$= -\frac{1}{3} - \frac{1}{3} = -\frac{2}{3}$$



+ve area

−ve answer

PROBLEM.

Remember F.T.C. only applies when $f(x)$ is continuous on $[a, b]$. (Here $[-1, 1]$.)

Not true here so the above is NOT VALID.

Actual answer: $\displaystyle\int_{-1}^{1} \frac{1}{x^4}\, dx$ D.N.E.

How about $\int_0^1 xe^{x^2}\,dx$?

Can't use known rules of finding antiderivatives.

5.5 Substitution Rule

How to find antiderivative of xe^{x^2} ?

Strategy To simplify a complicated integrand (the $f(x)$ in $\int f(x)\,dx$)

1. find a part of the integrand whose derivative appears as a factor in the integrand (up to a constant multiple)
2. Call this part u & rewrite everything in terms of u
3. Integrate (find antiderivative)
4. Rewrite everything in terms of x again.

Example Find $\int xe^{x^2}\,dx$.

Solution ① Notice that x^2 has derivative $2x$ which is a constant multiple different from factor x in xe^{x^2}.

(2) So set $u = x^2$. Then we have $\frac{du}{dx} = 2x$

i.e. $x = \frac{1}{2}\frac{du}{dx}$.

Substitute in so everything is in terms of u:

$$\int x e^{x^2}\, dx = \int \frac{1}{2} \frac{du}{\cancel{dx}} e^u \,\cancel{dx}$$

The magic is that we allow ourselves to "cancel" the dx as though it were a #

$$= \int \frac{1}{2} e^u \, du$$

to end

(3) We know how to integrate this!!

$$\int \frac{1}{2} e^u \, du = \frac{1}{2} \int e^u \, du = \frac{1}{2} e^u + C$$

(4) Rewrite in terms of x: $\int x e^{x^2}\, dx = \frac{1}{2} e^{x^2} + C$.

CHECK: $\frac{d}{dx}\left(\frac{1}{2} e^{x^2} + C\right) = 2x\left(\frac{1}{2} e^{x^2}\right) = x e^{x^2}$.

Chain Rule

This process "undoes"/"reverses" the Chain Rule.

Justification

If $f(x)$ has antiderivative $F(x)$

then $\frac{d}{dx}\left[F(g(x))\right] = F'(g(x)) \cdot g'(x)$ (Chain Rule)

$= f(g(x)) \cdot g'(x).$

In other words/notation: $\int f(\underbrace{g(x)}) g'(x)\, dx = F(g(x)) + C$

Substitute in $u = g(x)$

Set $u = g(x)$; then $g'(x) = \frac{du}{dx}$

$$\int f(u) \underbrace{\left(\frac{du}{dx}\right) dx} = F(u) + C$$

in other notation

$$= \int f(u) \underbrace{du}$$

So, they Must be the same.

"Cancellation" works.

To summarize:

The Substitution Rule

If $u = g(x)$, then $\int f(g(x)) g'(x)\, dx = \int f(u)\, du.$

Example Find $\int s^4 \sqrt{3s^5 - 1}\, ds$.

Solution $u = s^5$ or $u = 3s^5 - 1$

$$\frac{du}{ds} = 5s^4$$

substitute : $\displaystyle\int \underbrace{\frac{1}{5}\frac{du}{\cancel{ds}}}_{s^4} \sqrt{3u-1}\; \cancel{ds} = \underbrace{\int \frac{1}{5}\sqrt{3u-1}\, du}$

How to find?

Reset strategy : $v = 3u - 1$ $\quad \underbrace{\frac{dv}{du} = 3}$ ← factor (up to constant multiple)

Now that we know we can pretend the du, dx are like #s, we can do this kind of thing:

$dv = 3du$

$du = \frac{1}{3} dv$

in $\frac{1}{5}\sqrt{3u-1}$.

$$\underset{\text{substitute}}{=} \int \frac{1}{5}\sqrt{v}\,\overbrace{\left(\frac{1}{3}\,dv\right)}^{du}$$

$$= \frac{1}{15}\int \underset{v^{\frac{1}{2}}}{\sqrt{v}}\, dv = \frac{1}{15}\left(v^{3/2}\right)\left(\frac{2}{3}\right) + C$$

$$= \frac{2}{45} v^{3/2} + C$$

Turn back into a formula in terms of u → $= \dfrac{2}{45}(3u-1)^{3/2} + C$

Turn back into a formula in terms of s. → $= \dfrac{2}{45}(3s^5-1)^{3/2} + C.$

Remark Substituting in $u = 3s^5 - 1$ directly works in 1 go of the strategy! Try it!

We wanted $\displaystyle\int_0^1 \underbrace{xe^{x^2}}_{\hookrightarrow \text{ antiderivatives } \frac{1}{2}e^{x^2} + C.} \, dx.$

So how to find ~~them~~: the area/definite integral?

2 methods: ① Use to say if $f(x) = xe^{x^2}$

$$F(x) = \frac{1}{2} e^{x^2}$$

use F.T.C. Part II

$$\text{Answer} = F(1) - F(0). = \left[\frac{1}{2}e^{x^2}\right]_0^1.$$

② Definite Integral Version of Substitution Rule. T.B.C.